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Abstract

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MATHEMATICS

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A ONE-PRODUCT DYNAMIC MODEL IN THE PRESENCE OF INSTANTANEOUS TRANSFORMABILITY OF CAPITAL

1. Formulation. Consider an economic system in which one product is created, with part of it going to consumption and part to increasing fixed and circulating capital. Let $T(t)$ be the labor resources at our disposal at time t (we regard this function as given), and let $K(t)$ be the fixed capital at time t (the function sought). We shall characterize the possible production methods by a production function $U(K, T)$, which gives the amount of net product created by labor T using fixed capital K per unit of time. The function $U(K, T)$ is a positive homogeneous function of degree one:

$$U(\lambda K, \lambda T) = \lambda U(K, T).$$

Thus, having at time t labor $T(t)$ and fixed capital $K(t)$, we produce, per unit of time, $P(t) = U(K(t), T(t))$ units of output (national income). It is assumed that the function $U(K, T)$ is based on optimal methods. Under the assumption that effects can be added and that arbitrary linear combinations of methods are admissible, this leads to the requirement that the function $U(x, 1)$ $(0 \leq x < \infty)$ be concave downward (a necessary and sufficient condition). As regards the nature of consumption, we shall assume that its volume $V(t)$ is given; its size may also be determined in accordance with the parameters of the system, for example $V(t) = V[t, T(t), K(t), P(t)]$. Two typical hypotheses are:

a) consumption is proportional to labor resources

$$V(t) = aT(t);$$

b) consumption depends on the quantity of product produced, namely, a certain share $(1-\gamma)$ of the product is consumed, while the remaining part γ goes to accumulation.

In this case the development of the economy can be described by a differential equation for the function $K(t)$—the volume of fixed (and circulating) capital as a function of time:

$$dK/dt = P(t) - V(t) = U(K(t), T(t)) - V[t, K(t), T(t), P(t)]; \tag{1}$$

in particular, under hypotheses a) and b) it takes the form

$$dK/dt = U[K(t), T(t)] - aT(t), \tag{1a}$$

$$dK/dt = \gamma U[K, T]. \tag{1b}$$

The construction of equation (1), as well as (1a) and (1b), rests essentially on the hypothesis of instantaneous transformation of capital from one form into another, namely into that form which is optimal for the existing ratio of the volumes of fixed capital and labor resources at the given moment.

Optimization of the development of the system is carried out at each moment of time (differential optimization), since the volume of production is determined by the state of the system and its maximality is taken into account through the introduction of the production function; the state of the system also determines the size of consumption. This consideration corresponds to type I of one-product models indicated in (1). Here we have in mind studying equation (1) and certain characteristics of the model.

The magnitude of the rate of efficiency of capital investments is equal to $\partial U/\partial K$; as is known, under the conditions of the given model it characterizes the increase in the production of output per unit of time corresponding to a unit of additional capital investment ([2]).

2. Integrability in explicit form. We shall give cases in which equations (1) are integrable in explicit form. Introduce the change of variable

$$S(t) = K(t)/T(t). \tag{2}$$

Equation (1) takes the form

$$S' + \frac{T'}{T}S = U(S, 1) - \frac{V}{T}, \tag{3}$$

and equation (1b)

$$S' + \frac{T'}{T}S = \gamma U(S, 1). \tag{3b}$$

Consider the cases:

a) $T = T_0 e^{\delta t}$, $0 \le t < \infty$, δ is the demographic rate of growth, the function U is arbitrary;

b) $U(K,T) = cK + bT$;

c) $U(K,T) = cK^\alpha T^{1-\alpha}$ $(0 < \alpha < 1)$ (the Cobb-Douglas function);

d) $U(K,T) = K \ln T/K$ $(T \ge K)$.

Equation (1b) is integrable in explicit form in all cases a, b, c, d. Equation (1a) is integrable in explicit form in cases a, b, and d.

3. Formulas for the rate of efficiency. According to what has been said, the rate of efficiency is equal to

$$n_\mathrm{e} = \partial U(K,T)/\partial K.$$

Upon introducing the variable S (2), we obtain

$$n_\mathrm{e} = U'_S(S,1). \tag{4}$$

We have

$$\frac{dP}{dt} = \frac{d}{dt}\left[TU(S,1)\right] = T'U(S,1) + TU'_S(S,1)S'_t.$$

Hence, using (3), we find

$$\frac{dP}{dt} = T'U(S,1) + Tn_\mathrm{e}\left[U(S,1) - \frac{T'}{T}S - \frac{V(t)}{T}\right],$$

which gives

$$n_\mathrm{e} = \frac{dP/dt - T'U(S,1)}{TU(S,1) - T'S - V(t)} = \frac{\frac{1}{P}\frac{dP}{dt} - \frac{T'}{T}}{1 - \frac{T'}{T}\frac{K(t)}{P(t)} - \frac{V(t)}{P(t)}}. \tag{5}$$

In particular, in the case of equation (1), we have

$$n_\mathrm{e} = \frac{\frac{1}{P}\frac{dP}{dt} - \frac{T'}{T}}{\gamma - \frac{T'}{T}\frac{K}{P}}. \tag{6}$$

From formula (6), as also by direct calculation, one obtains the formula for n_e in the case of equation (1b) and $U(K,T) = aK^\alpha T^{1-\alpha}$, namely

$$n_{\text{e}} = \alpha P/K. \tag{6'}$$

4. Allowance for technical progress. Technical progress in equation (1) is taken into account in the simplest way if, for production, ···

taking the formula

$$P(t) = e^{\rho t} U(K, T),$$

we obtain

$$K' = e^{\rho t} U(K, T) - V(t), \tag{7}$$

where $V(t)$ is equal to:

a) $V(t) = aT(t)$;

b) $V(t) = (1-\gamma) e^{\rho t} U(K, T),\ 0 < \gamma < 1.$

We proceed to the construction of the efficiency norm. We have

$$\frac{dP}{dt} = \frac{d}{dt}\left[e^{\rho t} U(K, T)\right] = \rho e^{\rho t} U + e^{\rho t} \frac{d}{dt}\left[T(t) U(S; 1)\right] =$$

$$= \rho e^{\rho t} T U(S, 1) + e^{\rho t} T' U(S; 1) + e^{\rho t} T U'_S(S, 1) S'_t;$$

but from (7)

$$S' = e^{\rho t} U(S, 1) - \frac{T'}{T} S - \frac{V}{T}.$$

In this case $n_{\text{e}} = e^{\rho t} U'_S(S; 1)$, therefore we have

$$\frac{dP}{dt} = \rho e^{\rho t} T U(S, 1) + e^{\rho t} T' U(S, 1) + T n_{\text{e}} \left[e^{\rho t} U(S, 1) - \frac{T'}{T} S - \frac{V}{T}\right],$$

$$n_{\text{e}} = \frac{\frac{dP}{dt} - \rho P - e^{\rho t} T' U(S, 1)}{T\left(e^{\rho t} U(S, 1) - \frac{T'}{T} S - \frac{V}{T}\right)} = \frac{\frac{1}{P}\frac{dP}{dt} - \left(\rho + \frac{T'}{T}\right)}{1 - \frac{T'}{T}\frac{K}{P} - \frac{V}{P}}.$$

5. Allowance for physical and moral depreciation. The equation for revalued (actual), rather than nominal, fixed assets, denoted by $\overline{K}$, will be

$$\frac{d\overline{K}}{dt} = U(\overline{K}, T) - \delta\overline{K}(t) - V(t),$$

where δ is the share of assets lost as a result of physical depreciation, moral depreciation, and the nonconformity of the assets to the structure of the required output. For this case we apply the general formula (the presence of the term $\delta\overline{K}(t)$ is equivalent to the corresponding increase in consumption). Therefore

$$n_{\mathrm{e}} = \frac{\frac{1}{P}\frac{dP}{dt} - \frac{T'}{T}}{1 - \frac{V}{P} - \frac{\overline{K}}{P}\frac{T'}{T} - \delta\frac{\overline{K}}{P}}.$$

Taking the construction period into account, the equation should be written differently. Since an increment of future fixed assets is obtained, they must be revalued and moral amortization must be taken into account. With a period of freezing funds in construction equal to v years, if one assumes a smooth character of the functions participating in the equation, then the equation determining the change in $\overline{K}$ (revalued fixed assets) is approximately written as follows:

$$(1+\beta)^v \frac{d\overline{K}}{dt} = U(\overline{K}(t), T(t)) - \delta\overline{K}(t) - V(t);$$

$$\beta = \frac{1}{P}\frac{dP}{dt} \quad \text{or} \quad \beta = \frac{1}{P}\frac{dP}{dt} + \delta.$$

Consequently,

$$n_{\mathrm{e}} = \frac{\left(\frac{1}{P}\frac{dP}{dt} - \frac{T'}{T}\right)(1+\beta)^v}{1 - \delta\frac{\overline{K}}{P} - \frac{V}{P} - \frac{T'}{T}\frac{\overline{K}}{P}(1+\beta)^v}.$$

More precisely, one should proceed from the analysis of the equation with delay. The influence of economic depreciation can be studied on a model of type III in work ([1]).

6. Asymptotics

Consider equation (1b); assuming

$$\lim_{t\to\infty} \frac{T'}{T} = \lambda,$$

one can establish the following theorems:

Theorem 1. Let c be the root of the equation

$$\lambda x = \gamma U(x, 1), \qquad 0 < x < \infty;$$

K be a solution of equation (1b); then

$$\lim_{t\to\infty} \frac{K}{T} = c, \qquad \bar{n}_e = \lim_{t\to\infty} n_e = U'_x(c, 1).$$

Theorem 2. Let $\gamma U(x, 1) > \lambda x$ for $0 < x < \infty$; K be a solution of equation (1); then

$$\lim_{t\to\infty} \frac{\ln K}{t} = a\gamma, \qquad \lim_{t\to\infty} n_e = \lim_{t\to\infty} \frac{U(K, T)}{K} = a,$$

where

$$a = \lim_{x\to\infty} U'_x(x, 1).$$

In the case where technical progress is taken into account, the following holds.

Theorem 3. Let, in a neighborhood of the zero point,

$$t^{\alpha-1} U(1, t) = [c_0 + O(t)]$$

and let there exist limits for n_e and for $xU'_x(x, 1)/U(x, 1)$ as $x \to \infty$. Then, if K is a solution of equation (7), the following asymptotic formulas hold:

$$\lim_{t\to\infty} \frac{K}{e^{\frac{\rho}{1-\alpha} t} T} = \left[\frac{(1-\alpha)c_0\gamma}{\rho + (1-\alpha)\lambda}\right]^{1/(1-\alpha)},$$

$$\lim_{t\to\infty} n_e = \alpha \frac{\rho + (1-\alpha)\lambda}{(1-\alpha)\gamma}.$$

Theorems of types 1, 2, 3 can also be formulated and proved in the case of a model of type (1a).

7. Numerical example

To illustrate the formulas obtained, we give a numerical example, specifying certain initial data. We assume that in the given state of the economy: the rate of growth of national income

$$\frac{1}{P}\frac{dP}{dt} = 0.08;$$

$T'/T = 0.02$; $V/P = 0.7$. From (6), $n_e = 0.23$, i.e. 23%. Formula (6′) would give the same value if $\alpha = 0.46$ (according to data for the U.S. economy, $\alpha = 0.33$). Taking technical progress into account, with $\rho = 0.02$, $n_e = 15.4\%$. If depreciation is taken into account with $\delta = 0.03$, $n_e = 30\%$, if $\bar{K} = K$, and $n_e = 0.27$, if $\bar{K} = 0.75K$. If both are taken into account with the same ρ and δ, then $n_e = 20\%$. Finally, if the duration of construction is taken into account with $\nu = 2$, taking $\bar{K} = 0.75K$ and

$$\beta = \frac{1}{P}\frac{dP}{dt},$$

then $n_e = 0.32$.

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2. L. V. Kantorovich, V. L. Makarov, *The Application of Mathematics in Economic Research*, **3**, 1965, pp. 70–72.

Note: Figure translations are in progress. See original paper for figures.