$t(x,y)$-QUASICONFORMAL MAPPINGS WITH TWO TRIPLES OF CHARACTERISTICS

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Abstract

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$t(x,y)$-QUASICONFORMAL MAPPINGS WITH TWO TRIPLES OF CHARACTERISTICS

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In the note (1), a new definition was given of the concept of quasiconformality with one triple of characteristics, which is an extension of the usual concept of quasiconformality with one pair of characteristics, due to M. A. Lavrent' ev. Here we construct the concept of quasiconformality with two triples of characteristics, which in the elliptic case coincides with quasiconformality in the sense of B. V. Shabat (see (2)); we reveal the connection of the mappings under consideration with systems of differential equations of mixed type and establish their differentiability almost everywhere.

1°. **Mapping of the ring $K(z;\alpha,\beta,\gamma)$ onto the ring $K(w;\alpha_1,\beta_1,\gamma_1)$.** Fix two triples of real numbers α,β,γ and $\alpha_1,\beta_1,\gamma_1$, satisfying the conditions

$$\gamma\neq 0,\quad \gamma_1\neq 0,\quad \alpha+\gamma\geq 0,\quad \alpha_1+\gamma_1\geq 0,$$
$$\alpha\gamma-\beta^2=\alpha_1\gamma_1-\beta_1^2=-t,\quad |t|\leq 1. \tag{1}$$

In the plane xOy we construct the ring $K(z;\alpha,\beta,\gamma)^*$ of t-complex numbers of the form

$$z=x-\frac{\beta}{\gamma}y+i_t\frac{1}{\gamma}y,$$

where $i_t^2=t$; x,y are real numbers.

In the plane uOv we construct the ring $K(w;\alpha_1,\beta_1,\gamma_1)$ of numbers of the form

$$w=u-\frac{\beta_1}{\gamma_1}v+i_t\frac{1}{\gamma_1}v,$$

where $i_t^2=t$; u,v are real numbers.

Since $\alpha\gamma-\beta^2=\alpha_1\gamma_1-\beta_1^2$, the type of the rings in the planes xOy and uOv is one and the same: either both rings are elliptic, or hyperbolic, or parabolic.

Let $u = u(x, y)$, $v = v(x, y)$ be single-valued real functions defined in a domain $D \ni z_0$.

The mapping

$$w = f(z) = u(x, y) - \frac{\beta_1}{\gamma_1} v(x, y) + i_t \frac{1}{\gamma_1} v(x, y)$$

of elements $z \in D$ of the ring $K(z; \alpha, \beta, \gamma)$ into the ring $K(w; \alpha_1, \beta_1, \gamma_1)$ of elements w is called a ***t*-mapping with two triples of characteristics** α, β, γ and $\alpha_1, \beta_1, \gamma_1$.

Along with the concept of t-convergence (or angular convergence), $z \to z_0$, considered in ([1]), we introduce also the concept of tangential convergence,

$$z \xrightarrow{t} z_0.$$

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* For more details on the ring, see ([1]).

A sequence $\{z_n\}$, $z_n \in K$, is called **tangentially convergent** to $z_0 \in K$ if for every $\varepsilon > 0$ and every $\varphi > 0$ there exists $N(\varepsilon, \varphi)$ such that, for $n > N(\varepsilon, \varphi)$,

$$\|z_n - z_0\| < \varepsilon, \qquad |\arg(z_n - z_0)| > \varphi^*$$

A mapping $w = f(z)$ is called t-continuous at the point $z_0 \in D$ if

$$1) \text{ when } z \xrightarrow{t} z_0, \qquad f(z) \xrightarrow{t} f(z_0);$$

$$2) \text{ when } z \xrightarrow{t} z_0, \qquad f(z) \xrightarrow{t} f(z_0).$$

Remark 1. By the last definition we have somewhat narrowed the class of t-continuous mappings considered in ([1]). The t-continuous mappings in the new sense form a proper part of the mappings continuous in norm.

The definition of a t-differentiable mapping given in ([1]) remains the same, but we drop the requirement of t-continuity of its principal part $f_1(z, z_0)$.

We shall call a mapping $f(z)$ **differentiable** at the point z_0 if at this point the functions $u(x, y)$ and $v(x, y)$ are differentiable.

A single-valued mapping $w = f(z)$, continuous in norm in a neighborhood of z_0 and t-continuous at the point z_0, is called **regular** at the point z_0 if, in the case $\gamma\gamma_1 > 0$, it preserves the orientation and the disposition of pairs of sectors at z_0, or changes both the one and the other (see ([1])); and, in the case $\gamma\gamma_1 < 0$, it preserves the orientation and changes the disposition of pairs of sectors, or conversely.

Definition 1. $f(z)$ maps the infinitely small t-circle $E_n(a, \beta, \gamma; z_0)$ of the ring $K(z; a, \beta, \gamma)$ onto the infinitely small t-circle $E_{n,1}(a_1, \beta_1, \gamma_1; w_0)$, $w_0 = f(z_0)$, of the ring $K(w; a_1, \beta_1, \gamma_1)$, if $f(z)$ is single-valued and continuous in norm in a neighborhood of z_0, t-continuous at z_0, and for every φ, $0 < \varphi < \infty$, the condition is satisfied:

$$\lim_{n\to 0} \frac{\max\limits_{z\in E_n^{\varphi}} |f(z) - f(z_0)|}{\min\limits_{z\in E_n^{\varphi}} |f(z) - f(z_0)|} = 1,$$

where E_n^{φ} is the set of points $z \in E_n$ for which $|\arg(z - z_0)| \le \varphi$.

2°. t-quasiconformal mappings with two triples of characteristics. Consider a mapping

$$w = T(z): \qquad u = u(x, y), \qquad v = v(x, y)$$

of a domain D of the xOy-plane into the uOv-plane.

$$^{*} \quad \|z - z_0\| = \sqrt{\left(\Delta x - \frac{\beta}{\gamma}\Delta y\right)^2 + \left(\frac{1}{\gamma}\Delta y\right)^2}$$

$$\arg(z - z_0) =$$

$$= \begin{cases} \dfrac{1}{2\sqrt{t}} \ln\left|\dfrac{\gamma\Delta x - (\beta - \sqrt{t})\Delta y}{\gamma\Delta x - (\beta + \sqrt{t})\Delta y}\right|, & \text{if } t > 0, \\ \dfrac{\Delta y}{\gamma\Delta x - \beta\Delta y}, & \text{if } t = 0, \\ \dfrac{1}{\sqrt{-t}}\left[\operatorname{arc} tg\left(\sqrt{-t}\,\dfrac{\Delta y}{\gamma\Delta x - \beta\Delta y}\right) + \psi(x, y)\right], \quad \psi \in \{0, +\pi, -\pi\}, & \text{if } t < 0. \end{cases}$$

Definition 2. In the domain D there is given a distribution of characteristics α, β, γ and $\alpha_1, \beta_1, \gamma_1$, associated with T, if in D the functions

$$\alpha[x, y, u(x, y), v(x, y)], \quad \beta[x, y, u(x, y), v(x, y)],$$

$$\gamma[x, y, u(x, y), v(x, y)];$$

$$\alpha_1[x, y, u(x, y), v(x, y)], \quad \beta_1[x, y, u(x, y), v(x, y)],$$

$$\gamma_1[x, y, u(x, y), v(x, y)],$$

satisfying conditions (1), are defined.

Definition 3. Suppose that in the domain D there is given a distribution of characteristics α, β, γ and $\alpha_1, \beta_1, \gamma_1$, associated with T. The mapping T is called a t-quasiconformal mapping of the domain D with two triples of characteristics α, β, γ and $\alpha_1, \beta_1, \gamma_1$, if at every point of D and for every point $z \in D$ it maps an infinitesimal t-circle $E_n(\alpha, \beta, \gamma; z)$ regularly onto an infinitesimal t-circle $E_{n_1}(\alpha_1, \beta_1, \gamma_1; w)$.

Theorem 1. *In order that a one-to-one and continuous mapping T with nonzero Jacobian and t-differentiable with associated characteristics α, β, γ and $\alpha_1, \beta_1, \gamma_1$, be t-quasiconformal with the same characteristics and, at points where $t = 0$, preserve t-angles (see (1)), it is necessary and sufficient that it satisfy the system*

$$\alpha[x, y, u(x, y), v(x, y)]v_x + \{\beta[x, y, u(x, y), v(x, y)] - \beta_1[x, y, u(x, y), v(x, y)]\}v_y + \gamma_1[x, y, u(x, y), v(x, y)]u_y = 0, \tag{2}$$

$$\{\beta[x, y, u(x, y), v(x, y)] + \beta_1[x, y, u(x, y), v(x, y)]\}v_x + \gamma[x, y, u(x, y), v(x, y)]v_y - \gamma_1[x, y, u(x, y), v(x, y)]u_x = 0.$$

3°. Differentiability of t-quasiconformal mappings. In this section we shall deal with t-quasiconformal mappings of a domain D with two triples of characteristics continuously distributed in D: $\alpha(x, y), \beta(x, y), \gamma(x, y)$ and $\alpha_1(x, y), \beta_1(x, y), \gamma_1(x, y)$ (see (1)).

Definition 4. A t-mapping $f(z)$ of the circle $K(z; \alpha, \beta, \gamma)$ onto the circle $K(w; \alpha_1, \beta_1, \gamma_1)$ has a finite t-dilatation at the point $z_0 \in D$ if, for every φ, $0 < \varphi < \infty$,

$$\lim_{n \to 0} \sup_{E_n^\varphi} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = r(\varphi) < +\infty.$$

Lemma. *Under a t-quasiconformal mapping T of the domain D with characteristics α, β, γ and $\alpha_1, \beta_1, \gamma_1$, the t-dilatation is finite almost everywhere.*

Lemma 2. *Under a t-quasiconformal mapping T of the domain D with characteristics α, β, γ and $\alpha_1, \beta_1, \gamma_1$, the functions $u(z)$ and $v(z)$ have finite t-dilatation almost everywhere in D.*

Lemma 3. *In order that a real function $u(x, y)$, defined and continuous in the domain D, have a complete differential almost everywhere on a measurable set $E \subset D$, it is necessary and sufficient that in the domain D there exist such a distribution of characteristics $\alpha(x, y)$, $\beta(x, y)$, $\gamma(x, y)$ that the t-dilatation of the function $u(z)$ with these characteristics be finite almost everywhere on E**.

With the aid of Lemmas 1-3 the main result is easily proved:

Theorem 2. *A t-quasiconformal mapping T of the domain D with characteristics α, β, γ and $\alpha_1, \beta_1, \gamma_1$ is differentiable almost everywhere in D.*

Remark 2. A t-continuous mapping differentiable at the point $z_0 \in D$ is t-differentiable at that point.

* The finiteness of the t-dilatation under the mapping T must be regarded as a t-property of the mapping T (see (1)).

** This lemma is essentially Stepanov' s theorem in a formulation convenient for us.

4°. **Some remarks on systems of differential equations of mixed type.** Consider the system

$$a_1 u_x + b_1 u_y + c_1 v_x + d_1 v_y = 0, \qquad a_2 u_x + b_2 u_y + c_2 v_x + d_2 v_y = 0 \tag{3}$$

with coefficients depending on x, y, u, v and defined in a domain G of four-dimensional space. We shall assume that nowhere in G does the equality

$$a_1/a_2 = b_1/b_2 = c_1/c_2 = d_1/d_2$$

hold, and that the conditions

$$\begin{aligned} &1) \quad [AC] = [BD] = 0, \quad [AD] + [BC] = 0; \\ &2) \quad [CD] = [AB] = 0, \quad -[AD] + [BC] = 0 \end{aligned}$$

are not satisfied.

Here the notation used is

$$[XY] = x_1 y_2 - x_2 y_1.$$

Under these restrictions, system (3) can be reduced to the form (2), with coefficients α, β, γ and $\alpha_1, \beta_1, \gamma_1$ satisfying condition (1), by means of rotations of the coordinate systems xOy and uOv through angles $\pi/2$ and $\pi/4$, and also by reflections with respect to the coordinate axes. The latter means that the local properties of solutions of the system of mixed type have been considered by us in the most general form.

We have restricted ourselves to considering only quasilinear systems of mixed type. If, however, one assumes that the functions $u(x, y)$ and $v(x, y)$ have in D partial derivatives of first and of some higher orders, then Definitions 2 and 3 are generalized. The characteristics α, β, γ and $\alpha_1, \beta_1, \gamma_1$ may now depend on the partial derivatives of the functions u and v. Theorem 1 remains unchanged. However, the requirements of t-differentiability, t-quasiconformality, and others acquire a greater burden, and the substance of Theorem 1 under these conditions has not yet been considered by us. For quasilinear systems, the substance of the results raises no doubts.

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Note: Figure translations are in progress. See original paper for figures.