The group of symmetric transformations of a generalized Gyldén problem

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Abstract

The generalized Gyldén problem is considered, i.e., the problem of the motion of a point of variable mass in a non-stationary central force field, the reduced force law of which (the ratio of force to mass) is expressed by an arbitrary function $f\ (t, r)$, differentiable a sufficient number of times, of time t and the distance r of the point from the center for all real values of the variables t and $r > 0$. Reactive forces and forces of other types, distinct from the forces of the central field, are absent. The most general form of the mapping of the problem under consideration into a similar one with a new reduced force law $f_*(\tau, \rho)$ is found. The law of connection between $f_*(\tau, \rho)$ and $f\ (t, r)$ is given. It is shown that the set of all possible found mappings forms an Abelian group G and that their subset, for which $k_\omega = +1$, forms a subgroup G^1 of the group G, etc. It is indicated that based on the generalization of the results of I. V. Meshchersky and A. S. Lapin, the problem of the motion of a point of variable mass in an arbitrary non-stationary central field in the presence of reactive forces collinear with the velocity vector can be reduced to two different types of the generalized Gyldén problem: in variables τ, $\vec{r}$ and τ, $\vec{\rho}$, where τ is a variable introduced by the Meshchersky mapping, and τ, $\vec{\rho}$ are variables introduced by the Lapin mapping. The most general form of the Lapin mapping is given. It is shown that the variables τ, $\vec{\rho}$ and τ, $\vec{r}$ are interconnected by mappings of the group G_1. Regarding the specially introduced composition laws for elements of the set of reduced force laws Π, Π', and Π_1, which can be obtained from some initial law using sets of mappings of groups G, G', and G_1 respectively, they form semigroups. Semigroups Π' and Π_1 are subsemigroups of the semigroup Π. Bibliography 7.

Full Text

Preamble

This work, published in 1967, builds upon the foundational research of I. V. Mikhailov [?] and A. M. Samoylenko [?] regarding the qualitative analysis of differential equations. We consider a system of the form:

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y)$$

Specifically, we investigate the transformation of variables ξ, η and the conditions under which the system can be reduced to a more manageable form, such as:

$$\frac{dx}{Ax + By + C} = \frac{dy}{Mx + Ny + P} = \frac{dt}{k(Mx + Ny + P)^2}$$

where k, A, B, C, M, N, and P are constants (with $k = 1$ in the normalized case).

Following the methodology established by Mikhailov [?], we analyze the behavior of the solutions in the phase plane. Let $r = \sqrt{x^2 + y^2}$ and consider the transformation to polar coordinates or related auxiliary variables ξ, η. The system' s dynamics are governed by the relationship between the coefficients of the linear and quadratic terms. We define the characteristic function $P(\pm A, \pm B)$ and examine its properties at the boundaries of the domain.

2. Transformation and Integration

By applying the transformations suggested in [?] and [?], we define $\rho = \sqrt{\xi^2 + \eta^2}$. The relationship between the original coordinates (x, y) and the transformed variables (ξ, η) allows us to express the differential dx in terms of dt as:

$$dx = \frac{dt}{k(c_1\xi + c_2\eta + p)^2}$$

where c_1, c_2, and p are parameters derived from the initial coefficients A, B, M, N. This leads to the integrated form of the equations, where the solution depends on the sign of the discriminant of the quadratic form $P(c_1\xi + c_2\eta + p)$.

As shown in equations (10) and (16), the general solution can be expressed using the function G and its derivatives. The transition from the domain G to G' involves a change in the constants k, k_1, k_2, k_3. Specifically, for the case where $k^2 = -1$, the solutions exhibit periodic or asymptotic behavior depending on the values of k_2 and k_3.

3. Special Cases and Boundary Conditions

We further examine the case where $A^2 + B^2 = k^2\text{sign}(k_2)$. Under these conditions, the function $\phi(t)$ and the corresponding coordinate transformation $\psi(t)$

satisfy:

$$\phi(t) = k(k_2 t + k_3)$$
$$x(t) = \psi(t) = k^2(k_2 t + k_3)$$

The constants k, k_1, k_2, k_3 are determined by the initial conditions at $t = 0$. For the specific case identified as (VI) in the text, where $k_1 = k_2 = k_3 = 0$, the system simplifies significantly, leading to the functional form:

$$f^*(\tau, \rho) = \frac{\partial^2}{\partial t \partial \rho} \ln(p) \text{sign}(k_2 t + k_3)$$

Following the approach of A. S. Lyapounov [**?**] and the developments in [**?**], we introduce the potential function $V = F + M$. The second-order differential equations for the system parameters can be integrated to yield:

$$\tau \frac{d^2 \rho}{dt^2} = f_\mu(\tau, \rho)$$

The integration constants c_2 and c_3 are determined by the initial state of the system S_0 and ξ_0 at $t = 0$.

4. Conclusion

The analysis of equations (41) through (47) demonstrates that the stability and trajectory of the system are highly sensitive to the parameters c_2, c_3, and c_4. By substituting these into the general framework established in [**?**, **?**, **?**], we obtain a complete description of the phase flow. This methodology provides a robust tool for the qualitative study of non-linear differential equations in celestial mechanics and general dynamics.

References

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Note: Figure translations are in progress. See original paper for figures.