

The solution of a boundary value problem by the method of R -functions

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Abstract

The work is devoted to a topical issue related to the practical calculation of electrostatic fields. One of the authors of this article previously introduced functions that allow for the exact satisfaction of boundary conditions for domains of practically arbitrary shape. These functions, while being functions of ordinary continuous arguments, simultaneously possess a number of properties characteristic of the functions of the algebra of logic. The method is illustrated using an example of calculating electrostatic fields that were previously calculated by the method of separation of variables. Analysis of the numerical calculation shows that the advantage of the proposed method compared to the method of separation of variables is as follows: a) the required accuracy is achieved even with a small number of series terms (in this case $n = 3$), which significantly reduces the computer time spent compared to the method of separation of variables; b) the proposed method is applicable even when the method of separation of variables cannot be used; c) the mathematical apparatus of R -functions used can also be extended to solving waveguide problems with various geometric configurations. 1 illustration. 8 bibliographic references.

Full Text

Introduction

The development of methods for constructing coordinate functions in the R -function method, as proposed by V. L. Rvachev and colleagues [?, ?, ?, ?, ?], has significantly expanded the capabilities of mathematical modeling in complex domains. This approach allows for the analytical description of geometric objects of various shapes using logical operations translated into algebraic form. In particular, for a given set of boundaries $(L_1), (L_2), \dots, (L_n)$, one can construct a function that vanishes on these boundaries and satisfies specific conditions within the domain.

To describe the geometry, we utilize the R -conjunction operation. For two variables x and y , the R -conjunction is defined as:

$$x \wedge_{\alpha} y = \frac{1}{1 + \alpha} (x + y - \sqrt{x^2 + y^2 - 2\alpha xy}) \quad (1.1)$$

where $-1 < \alpha \leq 1$. This operation allows for the construction of a function $\Phi(x, y)$ that defines the boundary of a complex domain (L) composed of several segments (L_i). If each segment (L_i) is defined by the equation $\phi_i(x, y) = 0$, then the combined boundary can be represented as:

$$\Phi(x, y) = \phi_1 \wedge_{\alpha} \phi_2 \wedge_{\alpha} \dots \wedge_{\alpha} \phi_n = 0 \quad (1.4)$$

Construction of the Solution

Let us consider the construction of a function $\Phi_0(x, y)$ that takes prescribed values U_i on the corresponding boundary segments (L_i). Following the methodology described in [?], we define auxiliary functions $\psi_i(x, y)$ for $i = 1, 2, \dots, n$:

$$\begin{aligned} \psi_1(x, y) &= \phi_2 \cdot \phi_3 \dots \phi_n \\ \psi_i(x, y) &= \phi_1 \dots \phi_{i-1} \cdot \phi_{i+1} \dots \phi_n \\ \psi_n(x, y) &= \phi_1 \cdot \phi_2 \dots \phi_{n-1} \end{aligned} \quad (2.1)$$

These functions possess the property that ψ_i vanishes on all boundary segments except (L_i). Consequently, the normalized structure of the solution can be represented as:

$$\Phi_0(x, y) = \frac{\sum_{i=1}^n U_i \psi_i^m}{\sum_{i=1}^n \psi_i^m} \quad (2.2)$$

where m is an even integer. To refine the solution and satisfy the governing differential equations (such as the Poisson or Laplace equation), we seek the general solution in the form:

$$\Phi(x, y) = \Phi_0(x, y) + \omega(x, y) \sum_{k=1}^N c_k \chi_k(x, y) \quad (2.3)$$

In this expression, $\omega(x, y) = 0$ represents the equation of the entire boundary (L), and $\chi_k(x, y)$ are elements of a complete set of functions, such as polynomials:

$$1, x, y, x^2, xy, y^2, \dots \quad (2.4)$$

The coefficients c_k are determined using variational or weighted residual methods, such as the Ritz or Galerkin method, as detailed in [?].

Numerical Example

As an illustrative example, consider the domain $ABCD$ with boundary conditions $U = U_0$ on the segment DEF and $U = 0$ on the remaining parts of the boundary. The geometry is defined by the function:

$$\omega(x, y) = (a^2 - x^2)(b^2 - y^2) \quad (3.2)$$

The initial approximation $\Phi_0(x, y)$ is constructed using the R -functions ϕ_1 and ϕ_2 :

$$\Phi_0(x, y) = U_0 \frac{\phi_2}{\phi_1 + \phi_2} \quad (3.1)$$

where the boundary segments are defined as:

$$\begin{aligned} \phi_1(x, y) &= \frac{1}{2} [x - (x - a)^2(y^2 - b^2)^2 - \sqrt{x^2 + (x - a)^4(y^2 - b^2)^4 + 2\alpha x(x - a)^2(y^2 - b^2)^2}] \\ \phi_2(x, y) &= \frac{1}{2} [-x - (x + a)^2(y^2 - b^2)^2 - \sqrt{x^2 + (x + a)^4(y^2 - b^2)^4 - 2\alpha x(x + a)^2(y^2 - b^2)^2}] \end{aligned}$$

For the case $n = 3$ and $\alpha = 0.9$, we apply the Galerkin method to determine the coefficients c_k . The resulting system of algebraic equations is derived from the integral relations:

$$\iint_D (\nabla^2 \Phi) \omega \chi_j dx dy = 0 \quad (3.7)$$

Solving this system yields the following approximate solution for the potential distribution:

$$\Phi(x, y) = U_0 [\Phi_0(x, y) - 0.7547\omega x + 0.04635\omega xy^2 + 0.3701\omega x^3] \quad (3.8)$$

Comparison with numerical results obtained via finite difference methods shows a high degree of accuracy, with a maximum relative error of approximately 0.4%. This demonstrates that even with a small number of terms ($n = 3$), the R -function method provides a robust and efficient means of solving boundary value problems in domains with complex geometries.

Note: Figure translations are in progress. See original paper for figures.

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