ON THE THEORY OF ORLICZ SPACES GENERATED BY VARIABLE \(N\)-FUNCTIONS

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Abstract

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ON THE THEORY OF ORLICZ SPACES GENERATED BY VARIABLE N-FUNCTIONS

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1°. In the present article we describe some properties of Orlicz classes $K_M(G)$ and Orlicz spaces $L_M(G)$, which are constructed with the aid of so-called variable N-functions $M(x, w)$. The theorems proved here are a generalization of the corresponding results for Orlicz spaces $L_M(G)$ in the case when $\operatorname{mes} G < \infty$ and $M(x, w) = M(w)$ (see (^{1})), as well as of the results of the note (^{3}); here the known Δ_2-condition ((^{1}), p. 35) for N-functions and the $\widetilde{\Delta}_2$-condition introduced in (^{3}) are particular cases of the more general Δ_2'-condition (see Definition 2). We note that a complete exposition of the theory of N-functions and Orlicz spaces in the case when $M(x, w) = M(w)$, and G is a closed set of finite measure situated in n-dimensional Euclidean space, is available in (^{1}) (there one also finds detailed references to the literature on this question). The case $\operatorname{mes} G = \infty$ and $M(x, w) = M(w)$ was studied in the works ($^{4-6}$).

2°. Everywhere in what follows, by G we denote some space of points x with a σ-finite measure. We shall call G a **space with a continuous measure** if, for every set $\mathcal{E} \subset G$ of finite measure and every $\varepsilon > 0$, there exists a partition of $\mathcal{E}$ into such mutually disjoint sets $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_R$ ($R \geq 2$) that $\operatorname{mes} \mathcal{E}_j < \varepsilon$ ($j = 1, 2, \ldots, R$). By $L_1^+(G)$ we shall denote the set of nonnegative functions from $L_1(G)$.

Definition 1. A function $M(x, w)$, defined on the set $G \times (-\infty, \infty)$, will be called a **variable N-function** if, for each fixed $x \in G$, it is an N-function of the argument w on the interval $(-\infty, \infty)$ (see (^{1}), p. 16), and, for each fixed $w \in (-\infty, \infty)$, it is measurable on G as a function of the argument x. For every variable N-function $M(x, w)$, by $M^*(x, w)$ we shall denote the function which, for each fixed $x \in G$, is the complementary function with respect to the N-function $M(x, w)$ (see (^{1}), p. 22). It is easy to prove that $M^*(x, w)$ is also a variable N-function and that $M^{**}(x, w) = M(x, w)$. By $\mu(x, w)$ we shall denote the function which, for each fixed $x \in G$, is the right derivative of the N-function $M(x, w)$ with respect to the argument w, and by $\mu^*(x, w)$ the function which, for each fixed $x \in G$, is the right derivative of the N-function $M^*(x, w)$ with respect to the argument w.

Definition 2. We shall say that a variable N-function $M(x,w)$ **satisfies on** G **the** Δ_2'**-condition** if the inequality

$$M(x,2w) \leq f(x) + CM(x,w) \qquad (x \in G,\ -\infty < w < \infty),$$

is fulfilled, where $f(x) \in L_1^+(G)$, and C is some positive number.

Theorem 1. Let $M(x,w)$ be a variable N-function. Then the following three assertions a), b), and c) are equivalent: a) the inequality

$$w\mu(x,w) \leq f_0(x) + C_0 M(x,w) \qquad (w \geq 0,\ x \in G),$$

holds, where $f_0(x) \in L_1^+(G)$, C_0 is some positive number; b) the inequality

$$(1+\gamma)M^*(x,w) \leq \psi_0(x) + w\mu^*(x,w) \qquad (w \geq 0,\ x \in G),$$

holds, where $\psi_0(x) \in L_1^+(G)$, γ is some positive number; c) $M(x,w)$ satisfies on G the Δ_2'-condition.

Theorem 2. If the variable N-function $M(x,w)$ satisfies on G the Δ_2'-condition, then the inequality

$$M^*(x,kw) \leq k^{1+\gamma} M^*(x,w) + \psi(x) \quad (w \geq 0,\ x \in G,\ 0 \leq k \leq 1),$$

where $\psi(x) \in L_1^+(G)$, γ is some positive number.

For the case when $M(x,w) = M(w)$, $\operatorname{mes} G < \infty$, and $M(w)$ satisfies the Δ_2-condition, Theorem 2 was formulated in the article of Ya. B. Rutitskii (^{2}).

Definition 3. Let $M(x,w)$ be a variable N-function. By $K_M(G)$ we shall denote the totality of all such real measurable functions $w(x)$, defined on G, for which

$$\rho(w;M) = \int_G M(x,w(x))\,dx < \infty.$$

The set $K_M(G)$ will be called an **Orlicz class**.

Theorem 3. *In order that the variable N-function $M(x,w)$ satisfy on G the Δ_2-condition, it is sufficient that at least one of the following inequalities hold:*

a) $\displaystyle \operatorname{vrai\,sup}_{x\in G} \left(\sup_{w > f_0(x)} \frac{w\mu(x,w)}{M(x,w)} \right) < \infty,$ where $f_0(x) \geq 0$ and $f_0(x) \in K_M(G)$;

b) $\displaystyle \operatorname{vrai\,inf}_{x\in G} \left(\inf_{w > \psi_0(x)} \frac{w\mu^*(x,w)}{M^*(x,w)} \right) > 1,$ where $\psi_0(x) \geq 0$ and $\psi_0(x) \in K_{M^*}(G)$;

c) $\operatorname{vrai}\inf\limits_{x\in G}\left(\inf\limits_{w>\varphi_0(x)}\dfrac{w\mu^*(x,w)}{M^*(x,w)}\right)>1,$ where $\varphi_0(x)\ge 0$ and $\mu^*(x,\varphi_0(x))\in K_M(G)$.

Definition 4. Let $M(x,w)$ be a variable N-function. By $L_M(G)$ we shall denote the totality of all such real measurable functions $w(x)$, defined on G, for which

$$\|w\|_{L_M(G)}=\sup_{\rho(v;M^*)\le 1}\int_G |w(x)|\,|v(x)|\,dx<\infty. \tag{1}$$

The set $L_M(G)$ will be called an **Orlicz space**. As was noted in (3), $L_M(G)$, with the norm (1) defined on it, is a Banach space.

Definition 5. We shall say that a sequence of functions $w_n(x)\in L_M(G)$ $(n=1,2,...)$ **converges in mean** to the function $w_0(x)\in L_M(G)$ if

$$\lim_{n\to\infty}\rho(w_n-w_0;M)=0.$$

A set $\mathfrak{M}\subset K_M$ will be called **bounded in mean** if

$$\sup_{w(x)\in\mathfrak{M}}\rho(w;M)<\infty.$$

Definition 6. Let $M(x,w)$ be a variable N-function. Since G has σ-finite measure, there exists a sequence $\widetilde{G}_k$ $(k=1,2,...)$ of measurable sets of finite measure, nested one in another, such that

$$\bigcup_{k=1}^{\infty}\widetilde{G}_k=G.$$

For any pair of natural numbers m and l, evidently, there is a measurable set $G_{ml}\subset\widetilde{G}_l$ such that

$$\operatorname{mes}(\widetilde{G}_l-G_{ml})\le 2^{-(m+l)}$$

and on the set G_{ml} the function $M(x,m)$ is bounded. Put

$$G_k=\bigcup_{l=1}^{k}\left(\bigcap_{m=1}^{\infty}G_{ml}\right).$$

It is clear that

$$G_1 \subset G_2 \subset \dots \subset G_k \subset \dots, \qquad \operatorname{mes} G_k < \infty$$

$$(k = 1, 2, \dots) \quad \text{and} \quad \bigcup_{k=1}^{\infty} G_k = G.$$

We shall regard the sequence of sets G_k $(k = 1, 2, \dots)$ as fixed. We shall say that a function $w(x) \in \mathfrak{A}_M(G)$ if its values are only rational numbers, it is measurable, finite-valued, and there exists a natural number k_0, depending, generally speaking, on $w(x)$, such that $w(x) = 0$ if $x \notin G_{k_0}$. It is clear that

$$\mathfrak{A}_M(G) \subset L_M(G).$$

By $E_M(G)$ we shall denote the closure of the set $\mathfrak{A}_M(G)$ in the space $L_M(G)$ in the metric (1).

We now formulate a number of theorems on Orlicz spaces and classes.

Theorem 4. The inclusions $\mathfrak{A}_M(G) \subset E_M(G) \subset K_M(G) \subset L_M(G)$ hold, and the set $\mathfrak{A}_M(G)$ is dense in the Orlicz class $K_M(G)$ in the sense of convergence in the mean.

Theorem 5. For every functional $f \in (E_M(G))^*$ there exists, and moreover is unique, a function $v(x) \in L_{M^*}(G)$ such that the equality

$$f(w(x)) = \int_G v(x) w(x)\, dx \tag{2}$$

holds for all $w(x) \in E_M(G)$.

Theorem 6. If there exists a not more than countable collection G'_ν $(\nu = 1, 2, \dots)$ of measurable subsets of the set G such that, for any set $\mathcal{E} \subset G$ of finite measure,

$$\inf_\nu (\operatorname{mes}((\mathcal{E} - G'_\nu) \cup (G'_\nu - \mathcal{E}))) = 0,$$

then the space $E_M(G)$ is separable.

Theorem 7. Let the variable N-function $M(x, w)$ satisfy the inequality

$$M(x, kw) \le k\delta(k) M(x, w) + \psi(x)$$

$(x \in G,\ w > 0,\ 0 < k < k_0)$, where $\psi(x) \in L_1^+(G)$; $k_0 > 0$; $\delta(k)$ is a nonnegative function defined on the interval $(0, k_0]$ such that $\lim_{k \to 0} \delta(k) = 0$. Then the relation

$$\lim_{\|w\|_{L_M(G)}\to\infty} \frac{\rho(w;M)}{\|w\|_{L_M(G)}} = \infty \tag{3}$$

holds.

In the case where $\operatorname{mes} G < \infty$ and $M(x,w) = M(w)$, a necessary and sufficient condition for (3) to hold was obtained by Ya. B. Rutitskii in (^{2}).

Theorem 8. Let the variable N-function $M(x,w)$ satisfy on G the Δ_2'-condition. Then: a) convergence in norm in the space $L_M(G)$ coincides with convergence in the mean; b) $E_M(G) = K_M(G) = L_M(G)$; c) the general form of a linear functional on $L_M(G)$ is given by equality (2);

d)

$$\lim_{\|w\|_{L_{M^*}(G)}\to\infty} \frac{\rho(w;M^*)}{\|w\|_{L_{M^*}(G)}} = \infty;$$

e) every set $\mathfrak{M} \subset L_M(G)$ bounded in norm is also bounded in the mean.

Theorem 9. Let G be a space with a nonatomic measure and let the variable N-function $M(x,w)$ not satisfy on G the Δ_2'-condition. Then: a) the Orlicz class $K_M(G)$ is not a linear set (and consequently $E_M(G) \neq K_M(G)$, $K_M(G) \neq L_M(G)$, and $E_M(G) \neq L_M(G)$); b) there exists a functional $f_0 \in (L_M(G))^*$ for which there is no function $v(x)$ such that for all $w(x)$ from $L_M(G)$, $v(x)w(x) \in L_1(G)$ and the equality

$$f_0(w(x)) = \int_G v(x)w(x)\,dx$$

holds; in other words, (2) is not the general form of a linear functional on $L_M(G)$.

Theorem 10. In order that the Orlicz space $L_M(G)$ be reflexive, it is sufficient, and if G is a space with a nonatomic measure, also necessary, that each of the variable N-functions $M(x,w)$ and $M^*(x,w)$ satisfy on G the Δ_2'-condition.

Theorem 11. Let $M_1(x,w)$ and $M_2(x,w)$ be variable N-functions satisfying on G the Δ_2'-condition; let $\varphi(x,w)$ be a function satisfying the Carathéodory condition, and suppose the inequality

$$M_2(x,\varphi(x,w)) \leq f(x) + CM_1(x,w) \quad (x \in G,\ -\infty < w < \infty), \tag{4}$$

holds, where $f(x) \in L_1^+(G)$, and C is some positive number. Then the operator Φ, defined by the equality $\Phi w(x) = \varphi(x,w(x))$, maps $L_{M_1}(G)$ into $L_{M_2}(G)$ and is continuous.

Theorem 12. Let G be a space with a nonatomic measure; $M_1(x,w)$ and $M_2(x,w)$ be variable N-functions satisfying on G the Δ_2'-condition; let $\varphi(x,w)$ be a function satisfying the Carathéodory condition, and suppose $\varphi(x,w(x)) \in$

$K_{M_2}(G)$ for every function $w(x) \in K_{M_1}(G)$. Then the operator Φ, defined by the equality $\Phi w(x) = \varphi(x, w(x))$, maps $L_{M_1}(G)$ into $L_{M_2}(G)$, is continuous, and is bounded on every ball of the space $L_{M_1}(G)$, and for the function $\varphi(x, w)$ the inequality

$$M_2(x, \varphi(x, w)) \leq f(x) + CM_1(x, w) \quad (x \in G,\ -\infty < w < \infty),$$

holds, where $f(x) \in L_1^+(G)$, and C is some positive number.

Remark. Theorems 9 and 12 are proved on the basis of a result of M. A. Krasnosel' skii ((^{7}), p. 26).

The following theorem is a generalization of the well-known Vitali theorem (see (^{8}), p. 167).

Theorem 13. Let the variable N-function $M(x, w)$ satisfy the Δ_2'-condition on G. Let, further, $w_n(x)$ $(n = 1, 2, ...)$ be a sequence of functions from the space $L_M(G)$, and let $w_0(x)$ be some real-valued measurable and almost everywhere finite function on G. In order that $w_0(x) \in L_M(G)$ and

$$\lim_{n \to \infty} \|w_n(x) - w_0(x)\|_{L_M(G)} = 0,$$

it is necessary and sufficient that the following conditions be satisfied: a) the sequence $w_n(x)$ $(n = 1, 2, ...)$ converges to the function $w_0(x)$ in measure on every set of finite measure $\mathcal{E} \subset G$; b) the equalities

$$\inf_{\{\mathcal{E} : \operatorname{mes} \mathcal{E} < \infty\}} \left(\sup_{n \geq 1} \int_{G \setminus \mathcal{E}} M(x, w_n(x))\, dx \right) = 0,$$

$$\inf_{\delta > 0} \left(\sup_{\{\mathcal{E} : \operatorname{mes} \mathcal{E} < \delta\}} \left(\sup_{n \geq 1} \int_{\mathcal{E}} M(x, w_n(x))\, dx \right) \right) = 0.$$

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Note: Figure translations are in progress. See original paper for figures.

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