ON POSSIBLE FORMULATIONS OF THE PRINCIPLE OF MICROCAUSALITY AND SOME OF THEIR CONSEQUENCES

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Abstract

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ON POSSIBLE FORMULATIONS OF THE PRINCIPLE OF MICROCAUSALITY AND SOME OF THEIR CONSEQUENCES

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Recently a question of fundamental importance was raised ([1,2]) concerning the possibility of giving a physical justification for the use, in the axiomatic approach —in particular in the axiomatic approach of N. N. Bogolyubov and collaborators ([3])—of one or another space of generalized functions. As is known, in the axiomatic approach one usually restricts oneself to considering only generalized functions of moderate growth, and this is done without any physical justification. Since, however, the choice of one or another space of generalized functions can substantially affect the consequences obtained, which are in principle subject to experimental verification, this problem deserves the most careful attention.

In ([1]) it was shown that, under certain restrictions (to be stated below) imposed on the most general possible form of the function $F^{\mathrm{ret}}(x)$, its Fourier transform —the physical scattering amplitude of two spinless particles—

$$A(s,t) = \int d^4x\, F^{\mathrm{ret}}(x) \exp\left(i\, \frac{k+k'}{2}\, x\right) \tag{1}$$

as $s \to \infty$ in the upper s-half-plane is bounded by an arbitrary linear exponential. And from this alone, by virtue of crossing symmetry and the polynomial boundedness of the amplitude in the physical regions of the direct and crossed reactions, it follows, on the basis of the generalized Phragmén–Lindelöf maximum principle, that it is polynomially bounded at complex infinity and, consequently, that dispersion relations can be written. From this also follow, under very general assumptions, the Pomeranchuk theorem and various other asymptotic relations between the amplitudes of crossed processes ([4]).

In obtaining this result, which in fact again returns $F^{\mathrm{ret}}(x)$ to the space of generalized functions of moderate growth, the following three restrictions were imposed on $F^{\mathrm{ret}}(x)$. For simplicity of discussion we shall state them as applied to the asymptotic amplitude

$$\hat{T}(k) \equiv A^{\text{ret}}_{\infty}(s,t) = \int d\tau\, T(\tau) e^{ik\tau} \equiv (T(\tau), e^{ik\tau}), \tag{2}$$

introduced by N. N. Meiman ([4]) and found to be very fruitful.

First, it was assumed that the maximal class of generalized functions subject to consideration has the form

$$T(\tau) = \sum_{\nu=0}^{\infty} \int d\sigma_{\nu}(\tau_0)\, \frac{(-1)^{\nu}}{\nu!}\, \delta^{(\nu)}(\tau - \tau_0). \tag{3}$$

Second, it was assumed that $T(\tau)$ satisfies the principle of microcausality in Bogolyubov' s form, in the sense that integration in

of the right-hand side of (3) extend only to the right τ-half-axis. Finally, third, $T(\tau)$ had to satisfy a special principle of "absence of long-range action," the essence of which consists in forbidding the propagation of a signal $e^{ik\tau}$ from one arbitrary point $\tau \geqslant 0$ to any other.

In this article we shall leave the first assumption unchanged, discuss the principle of microcausality in more detail from the mathematical point of view, and try to dispense altogether with the rather rigid principle of "absence of long-range action." The point is that this essentially new principle does not follow from the usual, physically justified axioms of quantum field theory and has no clear physical content*. The main purpose of this article is to show that there exists a formulation–and a quite natural one–of the principle of microcausality that is considerably more liberal than the requirements of the principle of "absence of long-range action"; accepting it is sufficient both for writing dispersion relations and for proving various asymptotic relations between the amplitudes of crossed processes.

1. Thus, the generalized function $F^{\text{ret}}(x)$ entering the integral representation (1) is assumed to satisfy the principle of microcausality, which has an unconditionally profound physical meaning. A certain difficulty may arise only in giving this physical principle also a precise mathematical meaning, bearing in mind that $F^{\text{ret}}(x)$ is a generalized, not an ordinary, function. In the case when $F^{\text{ret}}(x)$ is a generalized function of moderate growth, this problem is solved trivially and, what is more important, unambiguously. But when we are dealing with a generalized function of a more general type (for example, with a generalized function of the form (3) in the one-dimensional case), this problem may prove–and indeed does prove–to have several mutually nonequivalent solutions.

The point is that the only requirement we can impose on possible strict formulations of this principle is, obviously, the requirement of regularity. It means that in the particular case when $F^{\text{ret}}(x)$ reduces to an ordinary (integrable) function, each of these formulations must be equivalent to the ordinary understanding of the vanishing of this function outside the upper half of the light cone. Each of the three strict formulations of the principle of microcausality given below, for simplicity as applied to $T(\tau)$, satisfies the requirement of regularity.

The **first formulation of the principle of microcausality** consists in the following: among the functions of the basic space M_f there must be functions $f^{(-)}(\tau)$, different from zero only on the left τ-half-axis, and for all such functions

$$(T(\tau), f^{(-)}(\tau)) = 0. \tag{4}$$

The requirement of regularity obliges one in this case to ensure that equality (4) holds for a sufficiently broad set of functions $f^{(-)}(\tau)$—namely, such that in the particular case when $T(\tau)$ reduces to an ordinary function, it would follow from equality (2) for all functions $f^{(-)}(\tau)$ from this set that $T(\tau < 0) = 0$. The set of such functions $f^{(-)}(\tau)$ may be called sufficient, and M_f, consequently, must contain at least one of the sufficient sets of functions $f^{(-)}(\tau)$. The set of all functions $f_k(\tau) = e^{ik\tau}$ for $\tau < 0$, $f_k(\tau) = 0$ for $\tau \geqslant 0$, with all real k, or else with k belonging to the upper k-half-plane, will certainly be a sufficient set.

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* To avoid misunderstanding, let us note that the terms "signal" and "absence of long-range action" are used here in a somewhat conventional sense. Recall that $x = x' - x''$, and $\tau = x_0 - \text{ex}$. Let us also recall that the principle of absence of long-range action in the usual sense certainly follows from the principle of microcausality.

The **II formulation of the principle of microcausality** consists in the requirement

$$\int d\tau\, T(\tau) e^{ik\tau} = \int_0^\infty d\tau\, T(\tau) e^{ik\tau} \neq \iint_\Gamma dU_\tau(\tau) e^{ik\tau}, \tag{5}$$

where $U_\tau(\tau)$ is an ordinary (not generalized, though not necessarily finite, differentiable) function, which may also depend polynomially on k, and the area Γ cannot be continuously contracted to an area that does not include any part of the left τ-half-plane (or indeed all of that half-plane). The equality entering into (5) means that all $d\sigma_\nu(\tau)$ are concentrated only on the right τ-half-plane. The inequality, according to the idea of the formulation, in fact gives this equality a nonfictitious meaning; for this purpose one uses the fact of the linear independence of the functions $P_1(k)e^{ik\tau_1}$ and $P_2(k)e^{ik\tau_2}$ as functions of k for different, real or complex, τ_1 and τ_2.

The **III formulation of the principle of microcausality** consists in requiring that, whatever finite interval (a, b) of the right τ-half-plane may be,

$$\int_a^b d\tau\, T(\tau) e^{ik\tau} \neq \iint_{\Gamma'} dU_\tau(\tau) e^{ik\tau}, \tag{6}$$

where the area Γ' cannot be continuously contracted to some finite interval of the same half-plane. The essential difference between the II and III formulations is that, if the II formulation forbids the propagation of the "signal" $e^{ik\tau}$ from the right τ-half-plane only into the left one, while nevertheless allowing this "signal" to leave into the complex τ-plane, then the III formulation forbids the departure of this "signal" anywhere at all from the real right τ-half-plane, and also to infinity, even along it.

As for the values of k for which the relations (5) and (6) are to hold and for which, consequently, the functions $e^{ik\tau}$ must belong to M_f, here there are two possibilities. First, these may be k with $\operatorname{Im} k > 0$. In this case the physical amplitude will be defined as the limiting value (2) as $\operatorname{Im} k \to +0$ and, together with $T(\tau)$, will be a generalized function. Second, these may be real k. In this case the physical amplitude will already be an ordinary function, but this is achieved at the cost of violating the symmetry of the τ- and k-spaces and of additional restrictions on the behavior of $d\sigma_\nu(\tau)$ as $\tau \to \infty$. The results that interest us here do not depend on which of these two possibilities is chosen.

2. Meiman ([1]) showed that acceptance of the principle of "absence of long-range action" leads to

$$\rho \equiv \lim_{\nu\to\infty} \sqrt[\nu]{|C_\nu|} = 0, \qquad C_\nu \equiv \int_0^\infty d\sigma_\nu(\tau) e^{ik\tau}, \tag{7}$$

whence follows the boundedness of the amplitude $\widetilde{T}(k)$ as $|k| \to \infty$ in the upper k-half-plane by an arbitrary linear exponential. Together with crossing symmetry and polynomial boundedness of the amplitude in the direct and crossed reactions, this ensures both the possibility of writing dispersion relations,* and their proof under very general assumptions,

* In fact, as shown in ([5]), for writing dispersion relations it is sufficient (not all of it is even necessary, as N. I

$$\lim_{R\to\infty} \frac{1}{R} \int_0^{2\pi} d\theta\, |\sin\theta|\, \ln^+ |\widetilde{T}(Re^{i\theta})| = 0. \tag{*}$$

various asymptotic relations between cross-amplitudes. It is not difficult to see that neither this formulation nor this proof can be secured by either formulation I or formulation II of the principle of microcausality.

Indeed, in order to satisfy the requirement of formulation I of this principle, it is sufficient to impose on all $d\sigma_\nu(\tau)$ the condition that they be concentrated only on the right τ-half-axis. In this case it is impossible to obtain any restrictions on the behavior of $d\sigma_\nu(\tau)$ as $\nu \to \infty$ (and, hence, any restrictions on ρ, contrary to the assertion contained in (2)). As for the requirements of formulation II of the principle, they will certainly be fulfilled if one chooses the class of generalized functions

$$T_{\tau_0}(\tau) = \sum_{\nu=0}^{\infty} C_\nu(\tau_0) \frac{(-1)^\nu}{\nu!} \delta^{(\nu)}(\tau - \tau_0), \tag{8}$$

obtained from (3) for $d\sigma_\nu(\tau) = C_\nu(\tau_0)\delta(\tau - \tau_0)d\tau$, for which

$$(T_{\tau_0}(\tau), e^{ik\tau}) = e^{ik\tau_0} \sum_{\nu=0}^{\infty} C_\nu(\tau_0) \frac{(ik)^\nu}{\nu!}, \tag{9}$$

and imposes on $C_\nu(\tau_0)$, as a function of ν, the condition

$$\rho(\tau_0) = \overline{\lim_{\nu\to\infty}} \sqrt[\nu]{|C_\nu(\tau_0)|} \leqslant \tau_0. \tag{10}$$

Within formulation III of the principle of microcausality we also cannot secure the fulfillment of condition (7). Nevertheless, it is not difficult to show that the amplitude $\tilde{T}(k)$ in this case still turns out to be bounded by an arbitrary linear exponential in the upper k-half-plane. In the particular case of the "concentrated" generalized function (8), this formulation permits a "smearing" of the "signal" only along the right real τ-half-axis over a finite distance, as a result of which the sum on the right-hand side of (9), as $|k| \to \infty$ in the upper k-half-plane, cannot grow faster than $e^{ik\tau_0}$ decreases, up to some linearly exponential or more strongly growing factor. Thus, formulation III of the principle of microcausality, but not I and not II, together with cross-symmetry and polynomial boundedness of the amplitude in the physical regions of the reactions, fully ensures both the possibility of writing dispersion relations and the possibility of obtaining various asymptotic relations between cross-amplitudes.

In conclusion, we emphasize that here we have restricted ourselves to the consideration of: 1) an elastic (in fact, binary) process and 2) the maximal class (3) of generalized functions. The first restriction, however, is inessential (see in this connection [6], where the difficult problem of completely removing the second restriction is also discussed).

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Note: Figure translations are in progress. See original paper for figures.