ON TESTING A MULTIDIMENSIONAL POLYNOMIAL HYPOTHESIS

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 519.281

MATHEMATICS

V. P. PALAMODOV

ON TESTING A MULTIDIMENSIONAL POLYNOMIAL HYPOTHESIS

(Presented by Academician Yu. V. Linnik on 19 III 1966)

Recall that a random vector $x \in R^p$ is said to be **distributed according to the normal law** $N(p, \xi, \sigma)$ if its density is equal to

$$\frac{1}{\sqrt{|\sigma|}(2\pi)^{p/2}} \exp\left(-\frac{1}{2}(x-\xi)'\sigma^{-1}(x-\xi)\right),$$

where σ is a certain symmetric positive definite matrix (the covariance matrix); $\xi = E(x)$ is the vector of means; $(x-\xi)'$ is the row transposed with respect to the column $(x-\xi)$. Fix some finite number of normal laws $N(p^\alpha, \xi^\alpha, \sigma^\alpha)$, $\alpha = 1, \dots, a$, and for each α consider a sample $x_1^\alpha, \dots, x_{n_\alpha}^\alpha \in N(p^\alpha, \xi^\alpha, \sigma^\alpha)$ of arbitrary size n_α. By a multidimensional polynomial hypothesis we mean a statement of the form*

$$\pi_1[\xi^\alpha, \sigma^\alpha] = \dots \pi_r[\xi^\alpha, \sigma^\alpha] = 0, \tag{1}$$

where $\pi_1, \dots, \pi_r$ are arbitrary polynomials in the quantities $\xi^1, \dots, \xi^a, \sigma^1, \dots, \sigma^a$. Note that the general multidimensional linear hypothesis (see (1)) is a special case of (1), when $\pi_1, \dots, \pi_r$ are linear in $\xi^1, \dots, \xi^a$ and do not depend on $\sigma^1, \dots, \sigma^a$. The problem considered in this note is to describe similar tests for the hypothesis (1) of a prescribed level λ, $0 \le \lambda \le 1$.

We shall need the following notation. Let $x_1, \dots, x_n$ be a sample from the normal law $N(p, \xi, \sigma)$. Put $x_i = (x_i^1, \dots, x_i^p)$. The quantities

$$s_{ij} = \sum_{k=1}^{n} x_k^i x_k^j, \qquad \bar{x}_i = \sum_{k=1}^{n} x_k^i$$

are sufficient statistics of the joint distribution of $x_1, \dots, x_n$. Consider the vector s, formed by the quantities s_{ij}, $1 \le i \le j \le p$, and the vector $\bar{x} = (\bar{x}_1, \dots, \bar{x}_p)$. Next introduce an analogous vector A, formed by the quantities $\frac{1}{2}(\sigma^{-1})_{ii}$ and $(\sigma^{-1})_{ij}$ with $1 \le i < j \le p$, and the vector $\theta = -\sigma^{-1}\xi \in R^p$. Note that the parameters ξ and σ are expressed in terms of θ and A in a fractional-linear way, while the density of the joint distribution of $x_1, \dots, x_n$ is equal to

$$C(\xi, \sigma) \exp(-(A, s) - (\theta, \bar{x})) h(s, \bar{x}),$$

where $C(\xi, \sigma)$ is a certain positive function; $h(s, \bar{x})$ is a certain nonnegative function whose support is equal to the image U of the mapping

$$R^{np} \ni (x_1, \dots, x_n) \mapsto (s, \bar{x}) \in R^{(p^2+p)/2+p}.$$

In the general case, when we have a finite set of normal laws $N(p^\alpha, \xi^\alpha, \sigma^\alpha)$, $\alpha = 1, \dots, a$, the joint distribution of the samples

---

* If y^α is the general element of the sequence $y^1, \dots, y^a$, then by $f[y^\alpha]$ we denote any function of $y^1, \dots, y^a$.

$(x_1^\alpha, \dots, x_{n_\alpha}^\alpha)$ has the following density (in the obvious notation):

$$C\,[\xi^\alpha, \sigma^\alpha] \exp\left(-\sum_\alpha (A^\alpha, s^\alpha) - \sum_\alpha (\theta^\alpha, \bar{x}^\alpha)\right) h[s^\alpha, \bar{x}^\alpha],$$

where the support of the function h is equal to $U = \underset{\alpha}{\times} U_\alpha$. Thus, the problem of describing level-λ cotests for hypothesis (1) consists in finding all functions $\psi[s^\alpha, \bar{x}^\alpha]$, defined on U, bounded between $-\lambda$ and $1 - \lambda$, such that

$$\int \psi[s^\alpha, \bar{x}^\alpha]\, h[s^\alpha, \bar{x}^\alpha] \exp\left(-\sum_\alpha (A^\alpha, s^\alpha) - \sum_\alpha (\theta^\alpha, \bar{x}^\alpha)\right) ds^\alpha d\bar{x}^\alpha = 0 \qquad (2)$$

for all A^α, θ^α satisfying system (1) (see [2, 3]). To obtain this description, consider the ideal I in the ring of all polynomials with complex coefficients in the variables A^α, θ^α, generated by all polynomials that vanish for all real A^α, θ^α satisfying (1). Let $p_1, \dots, p_m$ be a basis of least length in the ideal I.

Theorem. Suppose that $n_\alpha > 2p^\alpha$, $\alpha = 1, \dots, a$. Then:

I. Every sufficiently smooth cotest ψ with compact support belonging to $\operatorname{int} U$ can be written in the form

$$\psi = \frac{1}{h} \sum_1^m p_i [D_{s^\alpha} D_{\bar{x}^\alpha}] \varphi_i, \qquad (3)$$

where the φ are certain bounded functions with compact supports belonging to $\operatorname{int} U$.

II. If $m = 1$, then every cotest ψ can be written in the form

$$\psi = \frac{1}{h}\, p_1[D_{s^\alpha}, D_{\bar{x}^\alpha}]\varphi, \tag{4}$$

where φ is a function with support in U, belonging to L_2, with weight

$$\exp\left(-\varepsilon \sum_\alpha (|s^\alpha| + |\bar{x}^\alpha|)\right),$$

where $\varepsilon > 0$ is arbitrary.

It is obvious that any function of the form (3), respectively (4), satisfies condition (2). To demonstrate the idea of the proof, we shall establish the theorem in the special case when hypothesis (1) corresponds to the Behrens–Fisher problem, i.e. $a = 2$, $p^1 = p^2 = r = 1$, $\pi_1 = \xi^1 - \xi^2$. As is easily verified, in this case system (1) is written as follows: $A^1\theta^2 = A^2\theta^1$. Consequently, I is the ideal generated by all polynomials in $A^1, A^2, \theta^1, \theta^2$ that are multiples of the polynomial $P_1 = A^1\theta^2 - A^2\theta^1$. The corresponding differential operator in (4) is the ultrahyperbolic one

$$p_1 = \partial^2/\partial s^1 \partial \bar{x}^2 - \partial^2/\partial s^2 \partial \bar{x}^1.$$

For convenience put $\tau = (A^1, A^2, \theta^1, \theta^2)$ and $t = (s^1, s^2, \bar{x}^1, \bar{x}^2)$. Condition (2) is rewritten as

$$\int \psi'(t) \exp(-(\tau, t))\, dt = 0, \tag{5}$$

where $\psi' = \psi h$,

$$h(t) = (s^1 - (\bar{x}^1)^2)^{(n_1-3)/2}(s^2 - (\bar{x}^2)^2)^{(n_2-3)/2}$$

(it is enough to assume that $n_1, n_2 > 1$).

The function $\exp(e, t)$, where $e = (1, 1, 0, 0)$, obviously grows exponentially in U. Choose a fundamental solution $E(t)$ for the operator $p_1(D)$, satisfying the condition

$$F\left[E(t)\exp(e,t)\exp(-\varepsilon|t|)\right] \leq \frac{1}{|\tau| + 1} \quad \forall\, \varepsilon > 0, \tag{6}$$

where $F[...]$ is the symbol of the Fourier transform.* Hence it is seen that the general—

* The existence of such a fundamental solution follows from Theorem 3.1.1, Ch. II [4], which must first be applied to the operator $p_1(D-e)$, and the fundamental solution obtained multiplied by $\exp(-e, t)$.

function E decreases exponentially in U. Since the function $\psi' = \psi^h$ grows in U no faster than some power of $|t|$, the generalized function

$$\varphi(s) = (E(s+t), \psi'(t))$$

is defined. From (6) it is not hard to infer that in fact φ is an ordinary function belonging, together with its gradient, to the space L_2 with weight $\exp(-\varepsilon|t|)$ for any $\varepsilon > 0$. Since E is the fundamental solution for $p_1(D)$, we have $p_1(D)\varphi = \psi'$, which implies (4).

It remains to verify that $\operatorname{supp}\varphi \subset U$. Fix an arbitrary point $s \in U$. The function $E(s+t)$ decreases like $\exp(-(e,t)+\varepsilon|t|)$ for any $\varepsilon > 0$ and satisfies the homogeneous equation $p_1(D)E(s+t) = 0$ in a neighborhood of the convex set U. Therefore it follows from the results of (5) that it can be written in the form

$$E(s+t) = \int_N \exp(-(\tau, t))\,\mu, \tag{7}$$

where the measure μ is concentrated on the set N described by the conditions

$$A^1\theta^2 = A^2\theta, \qquad |e - \operatorname{Re}\tau| < \varepsilon, \qquad \tau = (A^1, A^2, \theta^1, \theta^2) \in C^4,$$

and is such that the integral in (7) converges in the space of generalized functions defined in a neighborhood of U, exponentially decreasing as $|t| \to \infty$. Here the number $\varepsilon > 0$ may be chosen arbitrarily small.

From the form of the set N it is not hard to see that each exponential in the integral (7) decreases in U like $\exp(\delta|t|)$ with some $\delta > 0$. Therefore we may perform the following substitution:

$$\varphi(s) = (E(s+t), \psi'(t)) = \int_N (\exp(-(\tau, t)), \psi'(t))\,\mu.$$

The scalar product $(\exp(-\tau, t), \psi'(t))$ is an analytic function of τ in the complex domain $|e - \operatorname{Re}\tau| < \varepsilon$ and, according to condition (5), vanishes for real τ satisfying the condition $A^1\theta^2 = A^2\theta^1$. Therefore it vanishes on the entire set N. Consequently, the right-hand side of (8) is equal to zero, as was required to prove.

In the general case the proof of the theorem is also based on the results of (5). Essential for the application of these results is the circumstance that, under the condition $n_\alpha > 2p^\alpha$, $\alpha = 1, \dots, d$, the set U is convex.

In conclusion I express my gratitude to Yu. V. Linnik for posing the problem, and also to A. M. Kagan for very valuable discussions of the questions touched upon in the note.

Moscow State University
named after M. V. Lomonosov

Received
2 III 1966

CITED LITERATURE

1. E. Lehmann, *Testing Statistical Hypotheses*, "Nauka," 1964.

2. Yu. V. Linnik, DAN, 161, No. 3, 520 (1965).

3. Yu. V. Linnik, *Statistical Problems with Nuisance Parameters*, "Nauka," 1966.

4. L. Hörmander, *Linear Differential Operators with Partial Derivatives*, Moscow, 1965.

5. V. P. Palamodov, Dissertation, Moscow State University, 1965.

Note: Figure translations are in progress. See original paper for figures.