INVESTIGATION OF INTERPOLATION PROCESSES CONSTRUCTED FOR AN EXTENDED SYSTEM OF P. L. CHEBYSHEV NODES

MATHEMATICS

1967

SovietRxiv

Abstract

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UDC 517.512.6

MATHEMATICS

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INVESTIGATION OF INTERPOLATION PROCESSES CONSTRUCTED FOR AN EXTENDED SYSTEM OF P. L. CHEBYSHEV NODES

(Presented by Academician S. N. Bernstein on 28 XI 1966)

1°. Let a matrix of numbers be given

$$-1 \le x_n^{(n)} < x_{n-1}^{(n)} < \dots < x_1^{(n)} \le 1, \qquad n = 1, 2, \dots \tag{m}$$

The function $\theta = \arccos x$, $-1 \le x \le 1$, maps the matrix (m) one-to-one onto the matrix

$$0 \le \theta_1^{(n)} < \theta_2^{(n)} < \dots < \theta_n^{(n)} \le \pi.$$

We shall call the point $\theta_k^{(n)}$ the image of the point $x_k^{(n)}$ on the unit semicircle. It has been known for a comparatively long time that the law of distribution of the images $\{\theta_l^{(n)}\}$ plays an important role in the theory of interpolation. Thus, for example, in order that the Lagrange interpolation process

$$L_n(f, x) = \sum_{k=1}^{n} f(x_k^{(n)}) l_k^{(n)}(x), \qquad l_k = l_k^{(n)}(x) = \frac{\omega_n(x)}{(x - x_k^{(n)})\omega_n'(x_k^{(n)})},$$

$$\omega_n(x) = \prod_{j=1}^{n} (x - x_j^{(n)}),$$

converge uniformly for every function $f(x)$ continuous on $[-1, 1]$, it is necessary that the inequalities

$$C_1/n \le \theta_{k+1}^{(n)} - \theta_k^{(n)} \le C_2/n, \qquad k = 0, 1, 2, \dots, n;$$
$$n = 1, 2, \dots; \quad \theta_0 = 0; \quad \theta_{n+1} = \pi, \tag{1}$$

hold, where the constants $C_i > 0$, $i = 1, 2$, do not depend on n ([1]). An analogous result is known ([1]) for the Hermite-Fejér interpolation process.

If the matrix (m) is such that inequalities (1) are satisfied, then we say that the nodes m are quasi-uniformly distributed.

According to the classical result of S. N. Bernstein-G. Faber, there is no such matrix of nodes (m) for which, for every $f \in C$, the relation*

$$L_n(f, x) \to f(x), \qquad n \to \infty \tag{2}$$

holds uniformly on $[-1, 1]$.

Therefore S. N. Bernstein ([2], p. 500) posed the problem of constructing such an interpolation process $\{A_n(f, x)\}_{n=1}^{\infty}$ which converges uniformly for every $f \in C$ and at the same time has the property that the ratio σ_n of the degree of the polynomial A_n to the number of its nodes can be made arbitrarily close to one. This problem was solved by S. N. Bernstein himself ([3]), who constructed the polynomials A_n in the following way.

For a given fixed natural number p, put

$$A_n(f, x) = \sum_{k=1}^{n} a_k^{(n)}(f) l_k(x), \tag{3}$$

where

$$a_k^{(n)} = f(x_k^{(n)}), \qquad k \not\equiv 0 \pmod{2p};$$

$$a_{2pt}^{(n)} = \sum_{j=1}^{p} f(x_{2p(t-1)+2j-1}^{(n)}) - \sum_{j=1}^{p-1} f(x_{2p(t-1)+2j}^{(n)}).$$

* C is the set of all functions continuous on $[-1, 1]$.

It is not hard to see that, for sufficiently large p, σ_n is arbitrarily close to unity. S. N. Bernstein ([3]) proved that, for the nodes of P. L. Chebyshev

$$x_k^{(n)} = \cos \frac{2k-1}{2n} \pi, \qquad k = 1, 2, \dots, n; \; n = 1, 2, \dots, \tag{4}$$

for every $f \in C$ the relation

$$A_n(f, x) \to f(x), \qquad n \to \infty, \qquad -1 \le x \le 1 \tag{5}$$

holds uniformly. The polynomials A_n of S. N. Bernstein are remarkable also because there exists ([4]) a very broad class of node matrices (m), including the nodes (4), for which, for every $f \in C$, the relation (5) holds uniformly. A natural question arises: what must the nodes be in order that, for every $f \in C$, the interpolation process of S. N. Bernstein converge uniformly?

In ([5,6]) it was proved that, for this, it is necessary that the nodes be quasi-uniformly distributed.* The question of whether this condition is sufficient remained open. Theorem 1 gives an answer to this question.

2°. By an **extended system of Chebyshev nodes** we shall mean the node matrix

$$x_0^{(n+2)} = 1, \quad x_{n+1}^{(n+2)} = -1, \quad x_k^{(n+2)} = \cos\frac{2k-1}{2n}\pi,$$

$$k = 1, 2, \ldots, n;\ n = 1, 2, \ldots. \tag{6}$$

It is obvious that these nodes are quasi-uniformly distributed.

Theorem 1. *The interpolation process (3) of S. N. Bernstein, constructed at the nodes (6) for the function $\mu(x) = (x + |x|)/2$, diverges at the point $x = 0$.*

We outline the proof. We shall assume that the polynomial A_n is constructed with $p = 1$. Then it is not hard to see that

$$A_n(\mu, 0) = x_0(l_0 + l_1) + \cdots + x_{2p}(l_{2p} + 2l_{2p+1}), \qquad n = 4p.$$

Hence, after simple calculations, we obtain

$$A_n(\mu, 0) = \frac{1}{2} + \frac{1}{n\cos\pi/2n} - \gamma_n,$$

$$\gamma_n = \frac{1}{n}\sum_{k=1}^{2p-1}{}' \frac{1}{x_k}\left(\frac{x_{k-1}}{\sqrt{1-x_k^2}} - \frac{x_k}{\sqrt{1-x_{k-1}^2}}\right).$$

The prime means that the summation is taken over odd k.

With the help of straightforward computations we obtain

$$A_n(\mu, 0) > \frac{1}{2} + \frac{1}{n\cos\pi/2n} - \frac{682}{495\pi} - \tau_n, \qquad \tau_n \to 0, \quad n \to \infty.$$

It follows that

$$\lim_{n\to\infty} A_n(\mu,0) > 1/11\pi.$$

Consequently, the process diverges at the point $x = 0$.

3°. Denote by $\mathfrak{M}_\tau$, $\tau \geq 1$, the set of all node matrices (m) for which the quantity

$$\lambda_n^{(\tau)} = \lambda_n^{(\tau)}(m) = \max_{-1\leq x\leq 1} \left(\sum_{j=1}^{n} |l_j(x)|^\tau \right)^{1/\tau}$$

is bounded, i.e. $\lambda_n^{(\tau)} \leq C$, where C depends only on (m) and τ. Among the numbers $\lambda_n^{(\tau)}$, the quantity $\lambda_n^{(1)}$, which is the Lebesgue constant of the Lagrange interpolation process, is of special interest. The quantity $\lambda_n^{(\tau)}$, as a rule, is difficult to estimate. It is easy to see that

$$\lambda_n^{(1)} \leq n^{1-1/\tau} \lambda_n^{(\tau)}. \tag{7}$$

Therefore an upper estimate of the quantities $\lambda_n^{(\tau)}$ is of interest.

Theorem 2. *In the case of the nodes (6) the inequality*

$$\lambda_n^{(2)} \leq \sqrt{7}. \tag{8}$$

* I.e., that the inequalities (1) be satisfied.

We outline the proof. For the nodes (6)

$$(\lambda_m^{(2)})^2 = \frac{(x^2+1)T_n^2(x)}{2} + \frac{T_n^2(x)}{n^2} \sum_{k=1}^{n} \frac{1-x^2}{(x-x_k)^2} - \frac{T_n^2(x)}{n^2} \left\{ \sum_{k=1}^{n} \frac{2x}{x-x_k} \right\}$$
$$+ \frac{(1+x)^2 T_n^2(x)}{2n^2} \sum_{k=1}^{n} \frac{1}{1-x_k} + \frac{(1-x)^2 T_n^2(x)}{2n^2} \sum_{k=1}^{n} \frac{1}{1+x_k} = S_0 + S_1 - S_2 + S_3 + S_4. \tag{9}$$

Since

$$\frac{T_n'(x)}{T_n(x)} = \sum_{k=1}^{n} \frac{1}{x-x_k}; \tag{10}$$

$$\frac{(T_n'(x))^2 - T_n''(x)T_n'(x)}{T_n^2(x)} = \sum_{k=1}^{n} \frac{1}{(x-x_k)^2},$$

we have

$$S_1 = 1 - \frac{\sin 2n\theta \cos\theta}{2n\sin\theta}. \tag{11}$$

Consequently, $0 \le S_1 \le 1$.

We pass to the estimate of S_2. From (9) it is seen that

$$S_2 = \frac{2xT_n(x)}{n}\sum_{k=1}^{n} l_k(x)\frac{(-1)^{k-1}}{\sqrt{1-x_k^2}},$$

where $\{l_k(x)\}_{k=1}^n$ are the fundamental polynomials of the Chebyshev nodes. Since

$$\sum_{k=1}^{n} l_k^2(x) \le 2, \qquad -1 \le x \le 1, \tag{7}$$

it follows that

$$|S_2| \le \frac{2x|T_n(x)|}{n}\sqrt{2}\left(\sum_{k=1}^{n}\frac{1}{1-x_k^2}\right)^{1/2}. \tag{12}$$

In view of the fact that (8)*

$$\sum_{k=1}^{n}\frac{1}{\cos^2\theta_k/2} = \sum_{k=1}^{n}\frac{1}{\sin^2\theta_k/2} = 2n^2, \qquad \theta_k = \frac{2k-1}{2n}\pi, \tag{13}$$

from (12) we obtain

$$|S_2| \le 2x|T_n(x)|\sqrt{2} \le 2\sqrt{2}. \tag{14}$$

The identities (13) lead to the equality

$$S_3 + S_4 = (1+x^2)T_n^2(x).$$

Thus,

$$S_0 + S_3 + S_4 = \frac{3(1+x^2)}{2}T_n^2(x) \le 3, \qquad -1 \le x \le 1. \tag{15}$$

From (11), (14), (15) follows (8).

Corollary 1. For the nodes (6),

$$\lambda_n^{(1)} \le \sqrt{7n}.$$

This assertion follows directly from Theorem 2 and inequality (7).

Corollary 2. If the best approximation $E_n(f)$ of the function f by a polynomial of degree n satisfies the inequality

$$E_n(f) \le C/n^{\alpha}, \qquad \alpha > 1/2,$$

then the Lagrange interpolation process constructed for the function f at the nodes (6) converges uniformly to f on $[-1, 1]$.

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* These equalities follow very simply from identity (10). In (8) they are proved differently.

Indeed, the inequality is known ((2), p. 258)

$$|f(x) - L_n(f, x)| \le (1 + \lambda_n^{(1)}) F_n(f).$$

Therefore Corollary 2 follows directly from Corollary 1.

From Corollary 2, in particular, it follows that the Lagrange interpolation process constructed at the nodes (6) for the function $f(x) = |x|$ converges uniformly on $[-1, 1]$, while the Hermite-Fejér interpolation process constructed for the same function and at the same nodes diverges at $x = 0$ (9).

4°. Corollary 2 can be strengthened. To this end we first establish a theorem:

Theorem 3. *Let the fundamental Lagrange polynomials* $\{l_j(x)\}$ *be constructed at the nodes (6), and*

$$M_n(x) = \sum_{j=0}^{n+1} |l_j(x)|.$$

Then the estimate

$$M_n(x) \le \frac{4}{\pi} |T_n(x)| \ln n + 20, \qquad n \ge 10, \quad -1 \le x \le 1 \tag{16}$$

is valid.

We outline the proof. By virtue of the symmetry of the nodes, we may assume that $0 \le x < 1$. Since $|l_0(x)| + |l_{n+1}(x)| \le 1$, $-1 \le x \le 1$, we shall estimate the function

$$\psi(x)=\sum_{j=1}^{n}|l_j(x)|.$$

If x is a node, then (16) is obvious. Therefore let us suppose that $x_{p+1}<x<x_p$. We have

$$\psi(x)=\sum_{j=1}^{p-2}|l_j(x)|+\sum_{j=p-1}^{p+2}|l_j(x)|+\sum_{j=p+3}^{n}|l_j(x)|\equiv S_1+S_2+S_3.$$

By Theorem 2,

$$|S_2|\le 4\sqrt{7}. \tag{17}$$

Let

$$r_1(t)=\frac{1}{(\cos t-\cos\theta)\sin t},\qquad 0\le t<\theta;$$

$$r_2(t)=\frac{1}{(\cos\theta-\cos t)\sin t},\qquad \theta<t<\pi;$$

$$\theta^{(+)}=\arccos\frac{x+\sqrt{8+x^2}}{4},\qquad \theta^{(-)}=\arccos\frac{x-\sqrt{8+x^2}}{4},\qquad 0\le x<1.$$

It is not difficult to verify that $r_1(t)$ decreases on the interval $(0,\theta^{(+)})$ and increases on $(\theta^{(+)},\theta)$; $r_2(t)$ decreases on $(\theta,\theta^{(-)})$ and increases on $(\theta^{(-)},\pi)$. With the aid of these properties of the functions $r_i(t)$, $i=1,2$, after some calculations we obtain

$$S_1+S_3\le\frac{|T_n(x)|}{\pi}\left(\ln\operatorname{ctg}\frac{\theta-\theta_{p-1}}{2}\ \operatorname{ctg}\frac{\theta_{p+2}-\theta}{2}\ \operatorname{cosec}\theta_1\ \operatorname{cosec}\theta_n+\right.$$

$$\left.+\cos\theta\ln\operatorname{tg}\frac{\theta_{p-1}}{2}\ \operatorname{tg}\frac{\theta_{p+2}}{2}\right)+2\sqrt{7}\qquad (x=\cos\theta).$$

Hence, together with (17), (16) follows.

Theorem 4. *The Lagrange interpolation process* $\{L_n(f)\}$*, constructed at the nodes (6) for a function* f *from the Dini-Lipschitz class* $(\lim\omega_f(\delta)\ln\delta\to 0)$*, satisfies relation (2) uniformly on* $[-1,1]$.

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Received
20 X 1966

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Note: Figure translations are in progress. See original paper for figures.