ON ADDITIVE FUNCTIONS OF SETS CONTINUOUS IN A DIRECTION

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.5

MATHEMATICS

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ON ADDITIVE FUNCTIONS OF SETS CONTINUOUS IN A DIRECTION

(Presented by Academician S. L. Sobolev on 10 III 1966)

We shall consider countably additive real functions of sets $\Phi(E)$, defined on Borel sets E of the n-dimensional cube Ω with sides parallel to the coordinate axes $x_1, x_2, \dots, x_n$. For an arbitrary E from n-dimensional Euclidean space R_n we shall put $\Phi(E) = \Phi(E \cap \Omega)$. By $W\Phi(E)$ we shall denote the total variation of the function Φ on E. Finally, together with the function $\Phi(E)$, we shall consider the function $\Phi^h(E) = \Phi(E + h)$, where h is a vector in R_n, and $E + h$ denotes the set E shifted as a rigid body by the vector h.

Classical results (see, for example, ([1])) assert the equivalence of the following three conditions:

1°. The function $\Phi(E)$ is absolutely continuous with respect to Lebesgue measure.

2°.
$$\lim_{|h|\to 0} W[\Phi^h - \Phi](\Omega) = 0 \tag{1}$$

(the function $\Phi(E)$ is continuous with respect to translation).

3°.
$$\Phi(E) = \int_E f(x)\,dx, \qquad x \in R_n. \tag{2}$$

The present note is devoted to countably additive functions possessing properties 1° or 2° only along certain coordinate axes. In this case, as we shall show, conditions 1° and 2° cease to be equivalent. Functions possessing these properties are, generally speaking, singular. However, for them one can obtain certain integral representations analogous to (2).

Let $m_k A$ denote the k-dimensional Lebesgue measure of the set A, and let $P_i A$ be the projection of the set A onto the axis x_i.

Definition 1. The function $\Phi(E)$ is called **absolutely continuous in the coordinate direction** x_i if $|\Phi(E)| \to 0$ as soon as $m_1 P_{iE} \to 0$.

Example 1. The function of plane sets

$$\varphi(E) = m_1 P_1(E \cap d) = m_1 P_2(E \cap d),$$

where d is the diagonal of the unit square, is an example of a function absolutely continuous in each coordinate direction.

The function $\varphi(E)$ is obviously singular. Thus, absolute continuity in each coordinate direction is insufficient for ordinary absolute continuity.

Let $P_{R_s} E$ denote the projection of the set E onto the subspace $R_s,\ s < n$.

Definition 2. The function $\Phi(E)$ is called **absolutely continuous with respect to the collection of coordinate directions** $x_1, x_2, \dots, x_s,\ s < n$, if $|\Phi(E)| \to 0$ as soon as $m_s P_{R_s} E \to 0$.

From absolute continuity with respect to the collection $x_1, x_2, \dots, x_s$ it obviously follows that there is absolute continuity in each of the directions $x_1, x_2, \dots, x_s$. The converse is not true.

Let $R_s^{\perp}$ denote the orthogonal complement of R_s, $s < n$, in R_n, and let BG be the totality of all Borel sets from some set $G \subset R_n$.

Theorem 1. In order that the function $\Phi(E)$ be absolutely continuous with respect to the totality of directions $x_1, x_2, \dots, x_s,\ s < n$, it is necessary and sufficient that one of the following conditions be fulfilled:

1) $\Phi(E) = 0$, if $m_s P_{R_s} E = 0$.
2) For sets E of the form $E = E_s \times E_s^{\perp}$, where $E_s \in BR_s$, and $E_s^{\perp} \in BR_s^{\perp}$, the representation holds

$$\Phi(E_s \times E_s^{\perp}) = \int_{E_s} f(E_s^{\perp}, x^s)\, dx^s,$$

where $x^s \in R_s$; $f(E_s^{\perp}, x^s)$, for almost all x^s, is countably additive in $E_s^{\perp}$ and, for all $E_s^{\perp} \in BR_s^{\perp}$, is summable in x^s.

It follows from 1) that from the set of singularities of a function $\Phi(E)$ absolutely continuous with respect to the totality of directions $x_1, x_2, \dots, x_s$, one may remove any set Q such that $m_s P_{R_s} Q = 0$, without this set ceasing to be a set of singularities.

Theorem 2. If $\Phi(E)$ is absolutely continuous with respect to the totality of directions $x_1, x_2, \dots, x_s$, then the functions $\overline{W}\Phi(E)$, $\underline{W}\Phi(E)$, and $W\Phi(E)$ will have the same property. For $W\Phi(E)$ the representation holds

$$W\Phi(E_s \times E_s^{\perp}) = \int_{E_s} Wf(E_s^{\perp}, x^s)\, dx^s.$$

Analogous representations hold for the upper and lower variations $\overline{W}\Phi(E)$ and $\underline{W}\Phi(E)$.

Let $x_s^\perp \in R_s^\perp$. Denote $\{x_s^\perp\} = x_s^\perp \times R_s$.

Definition 3. The function $\Phi(E)$ is called **continuous under shifts with respect to the totality of directions** $x_1, x_2, \dots, x_s$, if $|\Phi(E)| \to 0$ as soon as $m_s[\{x_s^\perp\} \cap E] \to 0$ simultaneously for all $\{x_s^\perp\}$.

Lemma. Let Q be a set from $B\Omega$ such that for any $\{x_s^\perp\}$ the measure $m_s[Q \cap \{x_s^\perp\}] = 0$. Then for almost all, in the sense of the measure m_s, vectors $h \in R_s$, the equality

$$\Phi^h(Q) = \Phi(Q + h) = 0$$

is valid.

Theorem 3. In order that the function $\Phi(E)$ be continuous under shifts with respect to the totality of directions $x_1, x_2, \dots, x_s$, $s < n$, it is necessary and sufficient that one of the following conditions be fulfilled:

1. The function $\Phi(E)$ is absolutely continuous with respect to the measure $\psi(E)$, defined by the equality

$$\psi(E_s \times E_s^\perp) = W\Phi(\Omega_s \times E_s^\perp) \cdot m_s E_s,$$

   where $\Omega_s = P_{R_s}\Omega$, and, consequently,

$$\Phi(E) = \int_E f(x)\, d\psi(E).$$

2. The function $\Phi(E)$ satisfies condition (1) only for vectors $h \in R_s$.
3. The function $\Phi(E)$ vanishes on any set $E \in BR_n$ such that $m_s[E \cap \{x_s^\perp\}] = 0$ for any $\{x_s^\perp\}$.

Thus, any of the properties 1, 2, and 3 listed in Theorem 3 may be taken as the definition of a function continuous under shifts.

Corollary 1. If $\Phi(E)$ is continuous under shifts with respect to the totality $x_1, x_2, \dots, x_s$, then it is absolutely continuous with respect to this totality.

The converse, as we have seen, is false.

Corollary 2. The projection onto $R_s^\perp$ of the set of singularities of a function $\Phi(E)$ that is continuous under shifts jointly in $x_1, x_2, \dots, x_s$ has $(n-s)$-dimensional measure zero.

Corollary 3. If the function $\Phi(E)$ is continuous under shifts jointly in $x_1, x_2, \dots, x_s$ and absolutely continuous jointly in $x_{s+1}, x_{s+2}, \dots, x_n$, then it is absolutely continuous in the usual sense.

Example 2. The function $\eta(E)$, defined for sets $E = E_s \times E_s^\perp$ by the equality

$$\eta(E) = \gamma(E_s) \cdot \delta(E_s^\perp),$$

is continuous under shifts jointly in $x_1, x_2, \dots, x_s$, if $\gamma(E_s)$ is an absolutely continuous set function, $E_s \in BR_s$, and $\delta(E_s^{\perp})$ is a countably additive set function, $E_s^{\perp} \in BR_s^{\perp}$.

The following example shows that the simultaneous fulfillment of Corollaries 1 and 2 is only a necessary condition for shift continuity of the function $\Phi(E)$.

Example 3. In the plane square $\Omega = \{0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ construct the graph of the "Cantor staircase" $y = f(x)$ and remove from this graph the intervals on which the function $f(x)$ is constant. Denote by C the remaining set of points on the graph, and define a set function $E \subset \Omega$ by the formula

$$\varphi(E) = m_1 P_y(E \cap C),$$

where P_y denotes projection onto the y-axis. The function thus constructed is, obviously, absolutely continuous in the y-direction, since $\varphi(E) = 0$ if $m_1 P_{yE} = 0$. The projection of its set of singularities (the set C) onto the x-axis has measure zero. At the same time, $\varphi(E)$ is not continuous under shifts along y.

The properties of set functions considered here are connected with differentiation in the generalized sense of S. L. Sobolev (see (2), and also (3)). We indicate two simple propositions:

1. If a countably additive set function $\Phi(E)$ has, as generalized derivatives in the directions $x_1, x_2, \dots, x_s$, also countably additive functions, then $\Phi(E)$ is continuous under shifts jointly in $x_1, x_2, \dots, x_s$.
2. If a countably additive function $\Phi(E)$ is absolutely continuous jointly in $x_2, x_3, \dots, x_n$, then as its primitive in the direction x_1 one may take a set function that is absolutely continuous in the usual sense.

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Received
1 III 1966

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Note: Figure translations are in progress. See original paper for figures.