EXTENSION OF DUAL SUBSPACES

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Abstract

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MATHEMATICS

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EXTENSION OF DUAL SUBSPACES

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Let Π be a Hilbert space in which, in addition to the ordinary scalar product $[x, y]$, an indefinite scalar product $(x, y) = [Jx, y]$ is given, where $J = P_+ - P_-$; P_+, P_- are mutually complementary orthoprojectors in Π.

Denote by $\mathfrak{M}_+$ the set of all maximal subspaces of $\mathfrak{P}_+ = \{x \in \Pi : (x, x) \geqslant 0\}$, and by $\mathfrak{K}_+$ the set of all operators acting in a nonexpanding manner from $\Pi_+ = P_+\Pi$ into $\Pi_- = P_-\Pi$, and analogously introduce the sets $\mathfrak{M}_-$, $\mathfrak{P}_-$, $\mathfrak{K}_-$. There are one-to-one correspondences $\mathfrak{M}_+ \leftrightarrow \mathfrak{K}_+$ and $\mathfrak{M}_- \leftrightarrow \mathfrak{K}_-$ ([1]). An algebra R of linear bounded operators in Π is called **symmetric** if from $A \in R$ it follows that $A^0 \in R$, where A^0 is the operator J-adjoint to A, defined by the equality $(Ax, y) = (x, A^0 y)$, $x, y \in \Pi$. If $A = A^0$ ($UU^0 = U^0 U = I$), then the operator A (U) is called J-self-adjoint (J-unitary). A pair of subspaces $\{\mathcal{L}_1, \mathcal{L}_2\}$ is called **dual** if $(\mathcal{L}_1, \mathcal{L}_2) = 0$ and $\mathcal{L}_1 \subset \mathfrak{P}_+$, $\mathcal{L}_2 \subset \mathfrak{P}_-$, and a maximal dual pair if, in addition, $\mathcal{L}_1 \in \mathfrak{M}_+$ and $\mathcal{L}_2 \in \mathfrak{M}_-$.

R. S. Phillips posed the problem of extending a dual pair of subspaces $\{\mathcal{L}_1^0, \mathcal{L}_2^0\}$, invariant with respect to an algebra R, to a maximal dual pair of subspaces $\{\mathcal{L}_1, \mathcal{L}_2\}$ invariant with respect to R ([2]). In the commutative case the problem was solved under the additional condition of symmetry of R with respect to ordinary adjunction, and in the noncommutative case under $A^0 = A^*$ for all $A \in R$, where A^* is the ordinary adjoint of A, or if $\mathcal{L}_1 \oplus \mathcal{L}_2 = \Pi$, which is equivalent in the commutative case to the fact that the group of J-unitary operators G generating the algebra R is bounded in norm by some constant C ([2]). Thus, for $U \in G$, $\|U^n\| \leqslant C$, $n = 0, \pm 1, \pm 2, \ldots$.

We shall solve the extension problem for a commutative algebra R under the condition $\|U^n\| \leqslant C_U$, $n = 0, \pm 1, \pm 2, \ldots$, and $P_+ U P_- \in \gamma_\infty$ for every $U \in G$, where γ_∞ is the aggregate of all completely continuous operators in Π.

Theorem 1. *Any dual pair of subspaces $\{\mathcal{L}_1, \mathcal{L}_2\}$, invariant with respect to a commutative algebra R, extends to a maximal dual pair of subspaces $\{\mathcal{L}_1, \mathcal{L}_1\}$ invariant with respect to R, if for every $U \in G$, $\|U^n\| \leqslant C_U$, $n = 0, \pm 1, \pm 2, \ldots$, and $P_+ U P_- \in \gamma_\infty$.*

Proof. The operator U, with respect to the decomposition $\Pi = \Pi_+ \oplus \Pi_-$, has the matrix representation

$$U \sim \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \tag{1}$$

and maps $\mathfrak{M}_+$ onto itself, inducing a fractional-linear transformation of $\mathfrak{K}_+$ onto itself

$$f_U(K) = (U_{21} + U_{22}K)(U_{11} + U_{12}K)^{-1}. \tag{2}$$

The set $\mathfrak{K}_+$ is bicompact in the weak operator topology, and the transformation (2) is continuous in this topology by virtue of the condition

$P_+UP_- \in \gamma_\infty$ ([3]). By the Schauder–Tikhonov principle, the transformation (2) has a fixed point K_0 in $\mathfrak{K}_+$, realizing a subspace $\mathcal{L}_0 \in \mathfrak{M}_+$ invariant with respect to the operator U, so that $\mathcal{L} = \{x \in \Pi : x = x_+ \oplus K_0 x_+,\ x_+ \in \Pi_+\}$.

Let F_U be the set of fixed points of the transformation (2). Since the transformation (2) is weakly continuous, F_U is a bicompact subset of $\mathfrak{K}_+$ in the weak topology. For the group G_k formed by any finite number of operators $U_1, U_2, \dots, U_k$ from G, the assertion of Theorem 1 holds, since, by commutativity, G_k is bounded by the constant $\alpha_k = C_1 C_2 \dots C_k$, where $\|U_i^n\| \le C_i,\ n = 0, \pm 1, \pm 2, \dots;\ 1 \le i \le k$.

Let $\{\mathcal{L}_1^0, \mathcal{L}_2^0\}$ be any dual pair invariant with respect to R, and let $\widehat{\mathfrak{M}}_+$ be the set of all subspaces from $\mathfrak{M}_+$ containing $\mathcal{L}_1^0$ and J-orthogonal to $\mathcal{L}_2^0$. It follows from (4) that the set of operators $\widehat{\mathfrak{K}}_+$ from $\mathfrak{K}_+$ corresponding to the set of subspaces $\widehat{\mathfrak{M}}_+$ from $\mathfrak{M}_+$ under $\mathfrak{M}_+ \leftrightarrow \mathfrak{K}_+$ is a convex bicompact set in the weak operator topology. The set $\widehat{\mathfrak{M}}_+$ is mapped by each operator U from G onto itself, which induces a transformation of $\widehat{\mathfrak{K}}_+$ onto itself of the form (2), continuous in the weak operator topology. As above, the set of fixed points $\widehat{\mathfrak{K}}_+^U$ of this transformation is a bicompact subset in the weak operator topology, realizing the set of subspaces $\widehat{\mathfrak{M}}_+^U$ from $\widehat{\mathfrak{M}}_+$ invariant with respect to the operator U. Since there exists a subspace $\mathcal{L}_1^k$ from $\widehat{\mathfrak{M}}_+$ invariant with respect to the group G_k, it follows that $\bigcap_{G_k} \widehat{\mathfrak{K}}_+^U \neq \emptyset$, which, together with the bicompactness of the sets $\widehat{\mathfrak{K}}_+^U$, gives that $\bigcap_{G_k} \widehat{\mathfrak{K}}_+^U \neq \emptyset$. The last relation shows that there exists a subspace $\mathcal{L}_1$ from $\widehat{\mathfrak{M}}_+$, invariant with respect to G. If now $\mathcal{L}_2$ is the J-orthogonal complement in Π to $\mathcal{L}_1$, then $\mathcal{L}_2 \in \mathfrak{M}_-$, $\mathcal{L}_2^0 \subseteq \mathcal{L}_2$, and, moreover, $\mathcal{L}_2$ is invariant with respect to G.

Thus, the pair $\{\mathcal{L}_1, \mathcal{L}_2\}$ is a maximal dual pair of subspaces extending the initial dual pair $\{\mathcal{L}_1^0, \mathcal{L}_2^0\}$, invariant with respect to the group G, and therefore also with respect to the algebra R, since R is the linear span of G.

Let us give some consequences of the theorem proved. Let $B(H_1, H_2)$ be the space of all bounded linear operators from the Hilbert space H_1 into the Hilbert space H_2; $\mathfrak{K}$ the unit ball in $B(H_1, H_2)$; S the surface of the ball $\mathfrak{K}$, i.e. $S =$

$\{K \in \mathfrak{K} : \|K\| = 1\}$, and let $f(K)$ be a transformation of $\mathfrak{K}$ into itself of the form

$$f(K) = (A + BK)(C + DK)^{-1}. \tag{3}$$

Consider in the space $H = H_1 \oplus H_2$ the operator U which, relative to the decomposition $H = H_1 \oplus H_2$, is given by the matrix

$$U \sim \begin{pmatrix} C & D \\ A & B \end{pmatrix}. \tag{4}$$

Put $J = P_1 - P_2$, where $P_{iH} = H_i$, $i = 1, 2$. Obviously, if the operator U is J-unitary in H, then the transformation (3) induced by it maps $\mathfrak{K}$ onto $\mathfrak{K}$ in such a way that S is mapped onto S.

Introduce in H the sets $\mathfrak{P}_+$, $\mathfrak{M}_+$ $(\mathfrak{P}_-, \mathfrak{M}_-)$ analogously to the corresponding sets in Π.

Corollary 1. Let $\mathfrak{U}$ be a finite commutative family of transformations of $\mathfrak{K}$ into itself of the form (3), and let each operator constructed from the corresponding transformation from $\mathfrak{U}$ by means of relation (4) be J-unitary in the space $H = H_1 \oplus H_2$. If each of the transformations $f_i(K)$ from $\mathfrak{U}$ has a fixed point K_i such that $\|K_i\| < 1$, then there exists a common fixed point K of the transformations $\mathfrak{U}$ such that $\|K\| < 1$.

Corollary 2. The assertion of Corollary 1 remains valid without the requirement that the family $\mathfrak{U}$ be finite, if the transformations $f(K)$ from $\mathfrak{U}$ are continuous in the weak operator topology. In this case the norm of the common fixed point

$K \leqslant 1$. For weak continuity of a transformation of the form (3), one may require the full continuity of the operator D.

Theorem 2. If the transformation $f(K)$ from $\mathfrak{U}$ has a fixed point K_0 such that $\|K_0\| < 1$, then only the following cases are possible:

a) K_0 is the unique fixed point of the transformation $f(K)$;

b) there exists an uncountable set of fixed points of the transformation $f(K)$, and among them there is necessarily a point on the set S.

Proof. Let $\mathcal{L}_0$ from $\mathfrak{M}_+$ be the invariant subspace of the operator U corresponding to the fixed point K_0. Since $\|K_0\| < 1$, the decomposition

$$\Pi = \mathcal{L}_0 \oplus \mathcal{F}_0, \tag{5}$$

is valid, where $\mathcal{F}_0$ is the subspace from $\mathfrak{M}_-$ (1) invariant with respect to the operator U. With respect to the decomposition (5), the operator U has the matrix representation

$$U \sim \begin{pmatrix} \widetilde{U}_{11} & 0 \\ 0 & \widetilde{U}_{22} \end{pmatrix}. \tag{6}$$

Introduce a new definite scalar product

$$[x, y]_1 = (x_+, y_+) - (x_-, y_-), \tag{7}$$

where $x_+, y_+ \in \mathcal{L}_0$, and $x_-, y_- \in \mathcal{F}_0$. Let $\widetilde{P}_1(\widetilde{P}_2)$ be the orthoprojector in H onto $\mathcal{L}_0(\mathcal{F}_0)$, and let $\widehat{\mathfrak{K}}$ be the totality of all operators acting contractively (with respect to the new norm) from $\mathcal{L}_0$ into $\mathcal{F}_0$. The new indefinite scalar product

$$(x, y) = [\widetilde{J}x, y]_1, \tag{8}$$

where $\widetilde{J} = \widetilde{P}_1 - \widetilde{P}_2$, coincides with the scalar product (x, y), whence it follows that there exists a one-to-one correspondence $\mathfrak{M}_+ \leftrightarrow \widehat{\mathfrak{K}}$, and therefore also $\mathfrak{K} \leftrightarrow \widehat{\mathfrak{K}}$, with $\widetilde{S} \leftrightarrow S$, where $\widetilde{S}$ is the surface of the ball $\widehat{\mathfrak{K}}$ (i.e., $\widetilde{S} = \{\widetilde{K} \in \widehat{\mathfrak{K}} : \|\widetilde{K}\|_1 = 1\}$). The operator U induces on the ball $\widehat{\mathfrak{K}}$ the linear transformation

$$f_U(\widetilde{K}) = \widetilde{U}_{22}\widetilde{K}\widetilde{U}_{11}^{-1}, \tag{9}$$

which proves the assertion of Theorem 2.

Let $f(K) \in \mathfrak{U}$. Define in the usual way the degree of the transformation $f(K)$: $f^n(K) = f(f^{n-1}(K))$, where $(n-1)$ is any natural number. Obviously, $f^n(K) \in \mathfrak{U}$.

Theorem 3. Let the operator constructed from $f(K)$ by means of relation (4) be J-unitary in $H = H_1 \oplus H_2$. If there exists a fixed point $\widetilde{K}_0$ of the transformation $f^n(K)$ such that $\|\widetilde{K}_0\| < 1$, then there also exists a fixed point K_0 of the transformation $f(K)$ such that $\|K_0\| < 1$.

Proof. It is obvious that the transformation $f^n(K)$ is induced by the operator U^n, where U is the operator in H defined above. Further, from $f^n(\widetilde{K}_0) = \widetilde{K}_0$ and from the fact that $\|\widetilde{K}_0\| < 1$, we have, by (2),

$$\|(U^n)^l\| \leqslant C, \qquad l = 0, \pm 1, \pm 2, \dots \tag{10}$$

The J-unitary operators $U, U^2, \dots, U^{n-1}$ are bounded, which together with (10) gives

$$\|U^m\| \leqslant C, \qquad m = 0, \pm 1, \pm 2, \dots \tag{11}$$

It follows from (2) that (11) is equivalent to the assertion of Theorem 3.

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Note: Figure translations are in progress. See original paper for figures.