

# ON SOME EMBEDDINGS OF ORDERED SEMIGROUPS

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**Abstract**

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*MATHEMATICS*

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## ON SOME EMBEDDINGS OF ORDERED SEMIGROUPS

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1. By an **order** we mean a partial order. All notions of semigroup theory that are not specially defined are used in the usual sense <sup>(1)</sup>. If  $x \in A$  or  $x$  is the empty symbol  $\emptyset$ , we shall write  $x \in A$ . If an ordered semigroup (o. semigroup)  $A$  is a subsemigroup of such an o. semigroup  $B$ , whose order induces on  $A$  the order of  $A$ , then one says that the o. semigroup  $A$  is a subsemigroup of the o. semigroup  $B$ , or that the o. semigroup  $A$  is **embeddable** in the o. semigroup  $B$ . We shall say that the o. semigroup  $A$  is **isomorphic** to the o. semigroup  $B$  if there exists an isomorphism of the semigroup  $A$  onto the semigroup  $B$  preserving the order. Further, by a binary relation  $\sigma$  of an o. semigroup  $A$  we mean an arbitrary binary relation on  $A$ , generally speaking distinct from the order of the o. semigroup  $A$ . For quasi-ordered semigroups (q.o. semigroups) we use analogous terminology.
2. A binary relation  $\sigma$  of an o. (q.o.) semigroup  $A$  is called a **relation of right potential divisibility** if the o. (q.o.) semigroup  $A$  is embeddable in an o. (q.o.) semigroup  $B$  in which, for every  $(a, b) \in \sigma$ , the element  $b$  divides  $a$  on the right. A subset  $M$  of an o. (q.o.) semigroup  $A$  is called **potentially left-invertible** if the o. (q.o.) semigroup  $A$  is embeddable in an o. (q.o.) semigroup  $B$  in which every  $a \in M$  is left-invertible.
3. In the present note necessary and sufficient conditions are found for a binary relation of an o. semigroup to be a relation of right potential divisibility, for a subset of an o. semigroup to be potentially left-invertible, and for an o. semigroup to be embeddable in an o. semigroup with left invertibility. Analogous results are obtained for q.o. semigroups.

Since every semigroup is trivially ordered, from the results obtained there follow the known results of works <sup>(2-4)</sup> on potential one-sided divisibility and invertibility of elements in semigroups <sup>(2,3)</sup> and on embeddings of semigroups in semigroups with one-sided invertibility <sup>(4)</sup>. All the results of the present note are obtained by a unified method, by using the construction of a certain factor semigroup of a semigroup that is the free product of the semigroup  $A$  and

infinite monogenic semigroups generated by the elements of a binary relation of the semigroup  $A$ . This construction, in particular, makes it possible to obtain by a unified method all the results of works <sup>(2-4)</sup>, which in those works were obtained by different methods. Apparently, potential properties in o. semigroups are considered here for the first time.

4. Let  $A$  be a q.o. semigroup with respect to the quasi-order  $\geq$ . We shall say that a binary relation  $\sigma$  of the q.o. semigroup  $A$  satisfies condition (1), and a subset  $M$  of the q.o. semigroup  $A$  satisfies condition (2), if

$$(\forall (a, c) \in \sigma, x, y \in A) [(cx = cy \rightarrow ax = ay) \wedge (cx > cy \rightarrow ax \geq ay)], \quad (1)$$

$$(\forall c, d \in M, x, y \in A, a \in A) [(cdx = cdy \rightarrow ax = ay) \wedge (cdx > cdy \rightarrow ax \geq ay)]. \quad (2)$$

In a right-cancellative ordered semigroup, every binary relation of right potential divisibility satisfies condition (1), and every potentially invertible-on-the-left subset (Sec. 2) satisfies condition (2). It is easy to show that a subset  $M$  of an ordered semigroup  $A$  satisfies condition (2) if and only if the Cartesian product  $A \times C$ , where  $C$  is the subsemigroup of  $A$  generated by  $M$ , satisfies condition (1).

5. In what follows, by  $A$  we mean an ordered semigroup with respect to the quasiorder  $\geq$ , by  $\sigma$  an arbitrary binary relation in  $A$  satisfying condition (1), and by  $B$  the semigroup that is the free product of the semigroup  $A$  and all infinite monogenic semigroups generated by elements  $(a, b) \in \sigma$ . If  $\sigma = A \times C$ , where  $C$  is a subsemigroup of  $A$ , then by  $\tau_1$  we denote the following binary relation in  $B$ :

$$(u, v) \in \tau_1 \leftrightarrow (\exists u_1 \in B, u_2 \in B, x, y \in A) [u = u_1 x u_2 \wedge v = u_1 y u_2 \wedge (\forall a \in A)(ax = ay)].$$

If  $\sigma \neq A \times C$  for any subsemigroup  $C$  of the semigroup  $A$ , then by  $\tau_1$  we shall mean equality in the semigroup  $B$ . Next, by  $\tau_2$  and  $\tau$  we denote the following binary relations in  $B$ :

$$(u, v) \in \tau_2 \leftrightarrow (\exists u_1, u_2 \in B, (a, c) \in \sigma) [u = u_1(a, c)u_2 \wedge v = u_1 a u_2], \quad (u, v) \in \tau \leftrightarrow (\exists u = u_1, u_2, \dots, u_n = v \in B) (\forall 1 \leq i \leq n-1) [u_i = u_{i+1} \vee (u_i, u_{i+1}) \in \tau_1 \cup \tau_2 \vee (u_{i+1}, u_i) \in \tau_1 \cup \tau_2].$$

The relation  $\tau$  is a stable equivalence of the semigroup  $B$ . We denote the factor semigroup of the semigroup  $B$  by  $\tau$  by  $\overline{B}$ ; the  $\tau$ -class of  $\overline{B}$  containing  $u \in B$  by  $\overline{u}$ , and the set of all  $\overline{a} \in \overline{B}$ , where  $a \in A$ , by  $\overline{A}$ .

6. Denote by  $\geq_{\tau_1}, \geq_{\tau}$  the following binary relations in the semigroup  $\overline{B}$  (Sec. 5):

$$\begin{aligned} \overline{u} \geq_{\tau_1} \overline{v} &\leftrightarrow (\exists u_1, u_2 \in B, a, c \in A)[\overline{u} = \overline{u_1 a u_2} \wedge \overline{v} = \overline{u_1 c u_2} \wedge \\ &\wedge (u_1 = \phi \rightarrow a \geq c) \wedge (\forall x \in A)(x a \geq x c)], \quad \overline{u} \geq_{\tau} \overline{v} \leftrightarrow (\exists u_1 = \\ &= u, u_2, \dots, u_n = v \in B)(\forall 1 \leq i \leq n-1)(\overline{u_i} \geq_{\tau_1} \overline{u_{i+1}}). \end{aligned}$$

The relation  $\geq_{\tau}$  is a stable quasiorder in the semigroup  $\overline{B}$ ; therefore  $\overline{B}$  is an ordered semigroup with respect to the quasiorder  $\geq_{\tau}$ . From the definition of  $\geq_{\tau}$  it follows that if  $a \geq c$  ( $a, c \in A$ ), then  $\overline{a} \geq_{\tau} \overline{c}$ .

7. Denote by  $\lambda$  the following binary relation in  $\overline{B}$ :

$$(\overline{u}, \overline{v}) \in \lambda \leftrightarrow (\overline{u} \geq_{\tau} \overline{v} \wedge \overline{v} \geq_{\tau} \overline{u}).$$

The relation  $\lambda$  is a stable equivalence of the semigroup  $\overline{B}$ . We denote the factor semigroup of the semigroup  $\overline{B}$  by  $\lambda$  by  $\overline{\overline{B}}$ ; the  $\lambda$ -class of  $\overline{\overline{B}}$  containing  $\overline{u} \in \overline{B}$  by  $\overline{\overline{u}}$ , and the set of all  $\overline{\overline{a}} \in \overline{\overline{B}}$ , where  $a \in A$ , by  $\overline{\overline{A}}$ . In the semigroup  $\overline{\overline{B}}$  introduce the following binary relation  $\geq_{\tau^*}$ :

$$\overline{\overline{u}} \geq_{\tau^*} \overline{\overline{v}} \leftrightarrow \overline{u} \geq_{\tau} \overline{v}.$$

The relation  $\geq_{\tau^*}$  is a stable order in the semigroup  $\overline{\overline{B}}$ ; therefore  $\overline{\overline{B}}$  is an ordered semigroup with respect to the ordering  $\geq_{\tau^*}$ . From the definitions of  $\geq_{\tau}, \geq_{\tau^*}$  it follows that if  $a \geq c$  ( $a, c \in A$ ), then  $\overline{\overline{a}} \geq_{\tau^*} \overline{\overline{c}}$ .

8. With respect to the semigroup  $\overline{\overline{B}}$  (Secs. 5, 6), as a result of rather complicated arguments, the following lemmas can be proved.

**Lemma 1.** If  $\overline{\overline{a}} = \overline{\overline{c}}$  ( $\overline{\overline{a}} \geq_{\tau} \overline{\overline{c}}$ ), where  $a, c \in A$ , then  $a = c$  ( $a \geq c$ ).

**Lemma 2.** If  $\sigma = A \times C$ , where  $C$  is a subsemigroup of  $A$ , and  $c u_1 = c u_2$  ( $\overline{c u_1} \geq_{\tau} \overline{c u_2}$ ), where  $c \in C, u_1, u_2 \in B$ , then  $\overline{u u_1} = \overline{u u_2}$  for any  $u \in B$ .

**Lemma 3.** If  $A$  contains no idempotents,  $\sigma = A \times A, \overline{u u_1} = \overline{u u_2}$  ( $\overline{u u_1} \geq_{\tau} \overline{u u_2}$ ), where  $u \in B, u_1, u_2 \in B$ , then  $\overline{v u_1} = \overline{v u_2}$  ( $\overline{v u_1} \geq_{\tau} \overline{v u_2}$ ) for any  $v \in B$ .

9. From Lemma 1 it follows that the mapping  $\varphi(a) = \overline{\overline{a}}$  ( $a \in A$ ) is an isomorphism of the q.o. semigroup  $A$  onto the q.o. subsemigroup  $\overline{\overline{A}}$  of the q.o. semigroup  $\overline{\overline{B}}$  (Sec. 6). According to Sec. 5, for any  $(a, c) \in \sigma$  we have  $\overline{\overline{(a, c)}} = \overline{\overline{a}}$ . If the semigroup  $A$  is ordered, then from the preceding, according to Sec. 7, it follows that  $\varphi$  is an isomorphism of the ordered semigroup  $A$  onto the ordered subsemigroup  $\overline{\overline{A}}$  of the ordered subsemigroup  $\overline{\overline{B}}$ . Hence, according to Secs. 2, 4, it follows that

**Theorem 1.** *In order that a binary relation  $\sigma$  of an ordered (q.o.) semigroup  $A$  be a relation of right potential divisibility, it is necessary and sufficient that  $\sigma$  satisfy condition (1) in  $A$ .*

10. Let  $A_i$  ( $i = 1, 2, \dots$ ) be q.o. semigroups with respect to quasi-orders  $\geq_i$ , such that each q.o. semigroup  $A_i$  is a subsemigroup of the q.o. semigroup  $A_{i+1}$ , and let  $D$  be the limiting semigroup (2) of the set of all semigroups  $A_i$ . Denote by  $\geq$  the following binary relation in  $D$ :

$$a \geq c \leftrightarrow (\exists i)(a, c \in A_i \wedge a \geq_i c).$$

The relation  $\geq$  is a stable quasi-ordering in the semigroup  $D$ ; therefore  $D$  is a q.o. semigroup with respect to the quasi-order  $\geq$ . We shall call  $D$  a **limiting** q.o. semigroup. Each q.o. semigroup  $A_i$  is a subsemigroup of the q.o. semigroup  $D$ . If all  $A_i$  are ordered, then  $D$  is ordered.

11. Let  $M$  be a subset of the q.o. semigroup  $A$  satisfying condition (2). Then, according to Sec. 4, the relation  $\sigma = A \times C$ , where  $C$  is the subsemigroup of  $A$  generated by  $M$ , satisfies condition (1) in  $A$ . From Sec. 9 and Lemma 2 we obtain the existence of such q.o. semigroups  $A_i$  ( $i = 0, 1, \dots$ ) that  $A_0 = A$ , each q.o. semigroup  $A_i$  is a subsemigroup of the q.o. semigroup  $A_{i+1}$ , in any  $A_k$  ( $k = 1, 2, \dots$ ) every equation  $xc = a$  ( $c \in C$ ,  $a \in A_{k-1}$ ) is solvable, and the relation  $\sigma_k = A_{k-1} \times C$  satisfies condition (1) in  $A_{k-1}$ . In the q.o. limiting semigroup  $D$  (Sec. 10) of the set of semigroups  $A_i$ , any equation  $xc = a$  ( $c \in C$ ,  $a \in D$ ) is solvable. If the semigroup  $A$  is ordered, then, applying the construction of the semigroup  $B$  (Sec. 7), analogously to Sec. 9 we obtain that the  $A_i$  are ordered; therefore  $D$  will be ordered. Hence, according to Secs. 2, 4, it follows that

**Theorem 2.** *In order that a subset  $M$  of an ordered (q.o.) semigroup  $A$  be potentially left invertible, it is necessary and sufficient that  $M$  satisfy condition (2) in  $A$ .*

12. If the q.o. semigroup  $A$  contains no idempotents and  $\sigma = A \times A$  satisfies condition (1), then, by Lemma 3, analogously to Sec. 11 we obtain the existence of such q.o. semigroups  $A_i$  ( $i = 0, 1, \dots$ ) that  $A_0 = A$ , each q.o. semigroup  $A_i$  is a subsemigroup of the q.o. semigroup  $A_{i+1}$ , in any  $A_k$  ( $k = 0, 1, \dots$ ) each  $a \in A_{k-1}$  is left invertible, and  $\sigma_k = A_k \times A_k$  satisfies condition (1). Then the limiting q.o. semigroup  $D$  (Sec. 10) of the set of semigroups  $A_i$  has left invertibility. If  $A$  is ordered, then  $D$  is ordered. Hence it follows that

**Theorem 3.** *In order that an ordered (q.o.) semigroup  $A$  without idempotents be embeddable in an ordered (q.o.) semigroup with left invertibility, it is necessary and sufficient that the relation  $A \times A$  satisfy condition (1).*

13. Let  $D$  be an ordered (q.o.) semigroup with respect to the order (quasi-order)  $\geq$ , embeddable in an ordered (q.o.) group; let  $N$  be a nonempty

set; let  $D_\alpha$  ( $\alpha \in N$ ) be a right ideal of the semigroup  $D$ ; let  $R$  be the set of all  $(\alpha, a)$ , where  $\alpha \in N$ ,  $a \in D_\alpha$ ; let  $R\alpha$  ( $\alpha \in N$ ) be the set of all  $(\alpha, a)$ , where  $a \in D_\alpha$ . With respect to the multiplication  $(\alpha, a)(\beta, b) = (\alpha, ab)$ , the set  $R$  is a semigroup,

and  $R_\alpha$  is a subsemigroup of  $R$ . Each subsemigroup  $R_\alpha$ , with respect to the binary relation  $\geq_\alpha$ ,

$$(a, a) \geq_\alpha (a, b) \leftrightarrow a \geq b,$$

is ordered (quasiordered). Denote by  $\Gamma$  (by  $\Gamma_1$ ) the class of all such o.(q.o.) constructed semigroups  $R$  whose order (quasiorder) induces on each subsemigroup  $R_\alpha$  the order (quasiorder)  $\geq_\alpha$ .

**Theorem 4.** *In order that an o.(q.o.) semigroup  $A$  with idempotents be embeddable in an o.(q.o.) semigroup with left reversibility, it is necessary and sufficient that the o.(q.o.) semigroup  $A$  be isomorphic to an o.(q.o.) semigroup of the class  $\Gamma$  (of the class  $\Gamma_1$ ).*

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*Note: Figure translations are in progress. See original paper for figures.*

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