ON SEPARABLE DIVISORS OF THE ORDER OF A GROUP

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ON SEPARABLE DIVISORS OF THE ORDER OF A GROUP

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§ 1. In the work (1), with the aid of the notion introduced there of a Π-separable group, we succeeded, for finite groups, in uniting Sylow' s classical theorem on the existence and conjugacy of subgroups with the well-known theorem of P. Hall, which has also already become classical, on subgroups of solvable groups whose order and index are relatively prime (2).

In the present work the separability condition is given a more general form, making it possible to find several new criteria for the existence and conjugacy of subgroups in finite groups, for which the order and index need not be relatively prime. In addition, in § 4 the main theorem on indexials from (4) is supplemented.

At a number of stages of the investigation in § 3 the method of indexials (4) could have been used; however, we have given the exposition an independent character.

§ 2. We shall give the notation used: $\mathfrak{G}$ is a finite group; $|\mathfrak{G}|$ is its order; $\mathfrak{E}$ is the identity subgroup of $\mathfrak{G}$; Π is some set of prime numbers (empty or not); $\Pi(n)$ is the set of all prime divisors of the natural number n; a Π-divisor a of a natural number n is a divisor a of the number n for which $\Pi(a) \subseteq \Pi$; a Π-Sylow divisor $|\mathfrak{G}|$ is the greatest Π-divisor of $|\mathfrak{G}|$; $a \nmid b$ means that the natural number a does not divide the natural number b; $\subset$ and $\subseteq$ are the signs of proper and non-proper inclusion of sets.

§ 3. We shall present the results obtained by us.

Definition 1. The **type of a subgroup** $\mathfrak{H}$ of a group $\mathfrak{G}$ with respect to some series of subgroups of it

$$\mathfrak{R}_0 \supseteq \mathfrak{R}_1 \supseteq \cdots \supseteq \mathfrak{R}_t$$

(where the equalities $\mathfrak{R}_0 = \mathfrak{G}$ and $\mathfrak{R}_t = \mathfrak{E}$ are not obligatory) will be called the sequence of numbers

$$|\mathfrak{H} \cap \mathfrak{R}_0|,\ |\mathfrak{H} \cap \mathfrak{R}_1|, \dots,\ |\mathfrak{H} \cap \mathfrak{R}_t|.$$

Two subgroups $\mathfrak{H}_1$ and $\mathfrak{H}_2$ of a group $\mathfrak{G}$ will be called **of the same type** with respect to the given series of subgroups of $\mathfrak{G}$ if the types of $\mathfrak{H}_1$ and $\mathfrak{H}_2$ with respect to this series coincide.

Definition 2. If p is a prime number, then the greatest $\{p\}$-divisor of the natural number n will be called the **primary part**, or **p-part**, of the number n and will be denoted by $(n)_p$. Obviously, 1 will also be a primary part of n (if one takes p not dividing n).

A subgroup whose order is a primary part of $|\mathfrak{G}|$ will, as usual, be called a **Sylow subgroup** of $\mathfrak{G}$.

Definition 3. If for a divisor m of the number $|\mathfrak{G}|$ one can indicate a series

$$\mathfrak{G} = \mathfrak{G}_0 \supseteq \mathfrak{G}_1 \supseteq \cdots \supseteq \mathfrak{G}_\mu = \mathfrak{E} \tag{1}$$

of normal divisors of the group $\mathfrak{G}$ such that each primary part of the number m is the order of a subgroup of some class of conjugate Sylow subgroups of some member of the series (1) corresponding to this primary part, then we shall call m an **almost Sylow divisor** of $|\mathfrak{G}|$ (with respect to the series (1)).

If, moreover, each set

$$\mathfrak{G}_{\nu-1} \setminus \mathfrak{G}_\nu, \qquad \nu = 1, 2, \dots, \mu,$$

has a nonempty intersection with subgroups of no more than one of all the classes of conjugate Sylow subgroups mentioned, then we shall call m a **separable divisor** of $|\mathfrak{G}|$ (with respect to the series (1)). This last condition will be called the **separability condition**.

It is obvious that every divisor a of the order of $\mathfrak{G}$ for which $(a, |\mathfrak{G}| \setminus a) = 1$, i.e. the $\Pi(a)$-Sylow divisor of $|\mathfrak{G}|$ (cf. (3)), will be an almost Sylow divisor of $|\mathfrak{G}|$ (with respect to the series $\mathfrak{G} \supseteq \mathfrak{E}$). If, in addition, $\Pi(a) \subseteq \Pi$, and $\mathfrak{G}$ is a Π-separable group ([1]), then a will be a separable divisor of $|\mathfrak{G}|$ with respect to any chief series of $\mathfrak{G}$.

Lemma 1. If m is a separable divisor of $|\mathfrak{G}|$ with respect to the series (1), then all distinct prime parts of m can be ordered in such a way as $\delta_1, \delta_2, \dots, \delta_{k+1}$, $k \geq 0$, that they will be the orders of Sylow subgroups, respectively, for the subgroups $\mathfrak{G}_{j_1} \supset \mathfrak{G}_{j_2} \supset \cdots \supset \mathfrak{G}_{j_{k+1}} = \mathfrak{E}$ chosen in this way from the series (1), while for $1 \leq i < k+1$ one will have $\delta_i \setminus |\mathfrak{G}_{j_i+1}|$, as a consequence of which the indicated Sylow subgroups will also be Sylow subgroups of Definition 3.

Definition 4. The system of subgroups $\mathfrak{G}_{j_1}, \mathfrak{G}_{j_2}, \dots, \mathfrak{G}_{j_{k+1}} = \mathfrak{E}$ of Lemma 1 will be called the **system of supporting subgroups of the separable divisor m of the order of** $\mathfrak{G}$, and for $1 \leq i < k+1$ we shall also put $\delta_i = p_i^{\omega_i}$.

Lemma 2. If m is a separable divisor of $|\mathfrak{G}|$ with the prime parts and supporting subgroups indicated in Definition 4, and if $h_1, h_2, \dots, h_\mu$ is the sequence of indices of the series (1), then every h_ν, $\nu > j_i$, $1 \le i < k+1$, which is divisible by p_i, is no longer divisible by any of the numbers $p_1, p_2, \dots, p_{i-1}$.

Lemma 3. Let m be a separable divisor of $|\mathfrak{G}|$ with the prime parts and supporting subgroups indicated in Definition 4. Let M_{j_i,p_i} be the set of all h_ν, $\nu > j_i$, $1 \le i < k+1$, that are divisible by p_i, and let M_0 be the set of all h_ν that have not fallen into any one of the sets M_{j_i,p_i}. Put, moreover, $h'_\nu = (h_\nu)_{p_i}$ if $h_\nu \in M_{j_i,p_i}$, and put $h'_\nu = 1$ if $h_\nu \in M_0$ (in particular, $h'_\nu = 1$ if $\nu \le j_1$). Then $m = h'_1 h'_2 \cdots h'_\mu$, and this representation of m will be called **prime**.

Lemma 4. Let $\mathfrak{H}$ be some subgroup of $\mathfrak{G}$. Then any subgroup of $\mathfrak{G}$ conjugate to $\mathfrak{H}$ in $\mathfrak{G}$ will be of the same type as $\mathfrak{H}$ with respect to the series (1).

Definition 5. A group $\mathfrak{G}$ having a series of normal divisors $\mathfrak{G} = \mathfrak{G}_0 \supseteq \mathfrak{G}_1 \supseteq \cdots \supseteq \mathfrak{G}_\mu = \mathfrak{E}$ with the sequence of indices $h_1, h_2, \dots, \dots, h_\mu$ will be called $(h_1, h_2, \dots, h_\mu)$**-dispersive**.

Theorem 1. Let m be a separable divisor of $|\mathfrak{G}|$ with respect to the series (1), with the supporting subgroups and prime representation indicated in Lemmas 1 and 3. Then:

1) $\mathfrak{G}$ has at least one subgroup $\mathfrak{M}$ of order m, having, with respect to the series (1), type $m, m/h'_1, m/h'_1h'_2, m/h'_1h'_2h'_3, \dots, 1$;
2) $\mathfrak{M}$ is $(h'_1, h'_2, \dots, h'_\mu)$-dispersive;
3) the Sylow subgroups of order $p_i^{\omega_i}$ from $\mathfrak{M}$ are contained in $\mathfrak{G}_{j_i}$;
4) all subgroups of $\mathfrak{G}$ of the same type as $\mathfrak{M}$ with respect to the series (1) are conjugate to $\mathfrak{M}$ in the group $\mathfrak{G}$.

It is not hard to see that if a is a divisor of the order of any group $\mathfrak{G}$ such that $(a, |\mathfrak{G}|/a) = 1$, then all subgroups of $\mathfrak{G}$ of order a will be of the same type with respect to any invariant (and even normal) series of $\mathfrak{G}$. If, moreover, $\mathfrak{G}$ is a Π-separable group and $\Pi(a) \subseteq \Pi$, then, taking into account the remark made above that such an a will be a separable divisor of $|\mathfrak{G}|$ with respect to any chief series of $\mathfrak{G}$, we see that special cases of Theorem 1 will be the theorems on the existence and conjugacy of sub-

groups of P. Hall ([2]) for solvable groups and of S. A. Chunikhin ([1]) for Π-separable groups.

We note the following special cases of Theorem 1.

Theorem 2. *If m is an almost Sylow Π-divisor of the order of a Π-separable (Π-solvable) group $\mathfrak{G}$, then $\mathfrak{G}$ has at least one subgroup of order m satisfying conditions 1)–4) of Theorem 1.*

Theorem 2 assumes an especially simple form for solvable groups.

Theorem 3. *If m is an almost Sylow divisor of the order of a solvable group $\mathfrak{G}$, then $\mathfrak{G}$ has at least one subgroup of order m satisfying conditions 1)–4) of*

Theorem 1.

We shall indicate one more generalization of Theorem 1. For this we shall need the following auxiliary

Theorem 4. *Let $\mathfrak{G}$ have a normal divisor $\mathfrak{G}_1$, and let p^α, $\alpha \geq 0$, be any power of a prime p dividing $|\mathfrak{G}/\mathfrak{G}_1|$. Suppose $\mathfrak{G}_1$ contains a subgroup $\mathfrak{M}$ of order m such that all subgroups of $\mathfrak{G}_1$ conjugate to $\mathfrak{M}$ in $\mathfrak{G}$ are conjugate to $\mathfrak{M}$ already in $\mathfrak{G}_1$. Then $\mathfrak{G}$ has at least one subgroup of order $p^\alpha m$ containing $\mathfrak{M}$ as a normal divisor.*

Theorem 5. *Let $\mathfrak{G}$ have a normal divisor $\mathfrak{G}_1$, and let p^α, $\alpha \geq 0$, be any power of a prime p dividing $|\mathfrak{G}/\mathfrak{G}_1|$. Let m be a separable divisor of $|\mathfrak{G}_1|$ with respect to some series of normal divisors of $\mathfrak{G}$ passing through $\mathfrak{G}_1$.*

Then $\mathfrak{G}$ has at least one subgroup of order $p^\alpha m$ containing, as a normal divisor, a subgroup $\mathfrak{M}$ of order m from $\mathfrak{G}_1$, existing in $\mathfrak{G}_1$ by Theorem 1.

§ 4. We note also the following additions to the main theorem on indices (Theorem 6 from ([4])), which also give criteria for the existence of subgroups.

We use here the notation and terminology of our paper ([4]), and also the following. Let $\mathfrak{N}_i/\mathfrak{G}_i$ of order $n_i f_i$, $i \in W$, for $i = \beta$, coincide with $\mathfrak{F}_\beta/\mathfrak{G}_\beta$, and for $i > \beta$ with the normalizer of the subgroup $\mathfrak{F}_i/\mathfrak{G}_i$ in the group $\mathfrak{G}_{i-1}/\mathfrak{G}_i$. From Theorem 6 of paper ([4]) there follows the existence, for each index $(h)_R = h = f_\beta f_{\beta+1} \dots f_\nu$, of at least one proper Sylow extension of the group $\mathfrak{G}$

$$(ch)_R = ch = c_\beta f_\beta c_{\beta+1} f_{\beta+1} \dots c_\nu f_\nu,$$

$\Pi(c_i) \subseteq \Pi(\bar{P}_{f_i})$, $i \in W$, and, consequently, also of a subgroup of order ch, where the coefficient $c_i = 1$ if $(\bar{P}_{f_i}, n_i) = 1$, $i \in W$. This result is supplemented by the following

Theorem 6. *Every index $(h)_R$ of the group $\mathfrak{G}$ has at least one proper Sylow extension. Among these extensions there exists at least one*

$$(ch)_R = c_\beta f_\beta c_{\beta+1} f_{\beta+1} \dots c_\nu f_\nu,$$

for which the coefficient c_i, $i \in W$, will be greater than or equal to 1 according as the number $(\bar{P}_{f_i}, n_i)$ is so.

An immediate consequence of Theorem 6 is

Theorem 7. *If $\mathfrak{G}/\mathfrak{G}_1$ has a subgroup of order a, then $\mathfrak{G}$ contains at least one subgroup of order aa_1, $\Pi(a_1) \subseteq \Pi(a)$, whose intersection with $\mathfrak{G}_1$ is a special (i.e. nilpotent) subgroup of order a_1. Among these subgroups there exists at least one $\mathfrak{H}$ for which the number a_1 will be greater than or equal to 1 according as the number $(a, |\mathfrak{G}_1|)$ is so.*

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Note: Figure translations are in progress. See original paper for figures.