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A MODEL OF THE VERTICAL STRUCTURE OF A TIDAL FLOW IN A HOMOGENEOUS SEA COVERED WITH ICE

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The question of the structure of a tidal flow in a sea with a free surface has been studied theoretically rather well ([1–7]); however, the structure of a tidal flow in a sea covered with ice has been studied insufficiently. Apparently, the first work in which this question is touched upon is the work of Sverdrup ([1]), devoted to the processing of the results of the expedition on the vessel "Mod."

Let us consider the case of a homogeneous sea covered with ice. We shall represent the ice cover in the form of a flexible film, which reacts to the vertical motions of the fluid (takes the form of the wave passing beneath it) and at the same time exerts a retarding influence on horizontal motions. During the motion of a tidal wave under ice, owing to the adhesion of the fluid, the formation of two boundary layers must occur: at the bottom and at the lower edge of the ice cover. If the sea depth is small, then the upper and lower boundary layers may close, and then the entire thickness of the sea will be encompassed by turbulent mixing.

In this case, as a first approximation, it is natural to take the coefficient of turbulence k as constant within the entire thickness of the sea. In addition, it is expedient to assume that k represents a certain coefficient of turbulence averaged over the tidal period, whose magnitude must be found in the course of solving the problem.

Then the system of equations and boundary conditions describing the distribution of the velocity of the tidal current in the sea under ice will be written in the form

$$\partial u/\partial t - \lambda v = -g\,\partial\zeta/\partial x + k\,\partial^2 u/\partial z^2; \quad (1)$$

$$\partial v/\partial t + \lambda u = -g\,\partial\zeta/\partial y + k\,\partial^2 v/\partial z^2; \quad (2)$$

$$z = 0: \quad u = v = 0; \tag{3}$$

$$z = D: \quad u = v = 0, \tag{4}$$

where u, v are the components of the velocity of the tidal current; ζ is the oscillation of the sea level; λ is the Coriolis parameter; g is the acceleration of gravity; t is time; D is the depth; the z-axis is directed vertically upward; the origin of coordinates is located in the plane of the bottom.

For convenience in solving the problem, we reduce the system (1)–(4) to the form

$$\partial W/\partial t + i\lambda W = -g\,\partial\zeta/\partial n + k\,\partial^2 W/\partial z^2; \tag{5}$$

$$\partial W^*/\partial t - i\lambda W^* = -g\,\partial\zeta^*/\partial n + k\,\partial^2 W^*/\partial z^2; \tag{6}$$

$$z = 0: \quad W = W^* = 0; \tag{7}$$

$$z = D: \quad W = W^* = 0, \tag{8}$$

where

$$W = u + iv, \quad W^* = u - iv,$$

$$\partial\zeta/\partial n = \partial\zeta/\partial x + i\,\partial\zeta/\partial y, \quad \partial\zeta^*/\partial n = \partial\zeta/\partial x - i\,\partial\zeta/\partial y. \tag{9}$$

In a tidal flow the velocity components and the level are harmonic functions of time; consequently, one may write

$$W = \overline{W}e^{-i\sigma t}, \quad W^* = \overline{W}^* e^{-i\sigma t},$$

$$\zeta = \overline{\zeta}e^{-i\sigma t}, \quad \zeta^* = \overline{\zeta}^* e^{-i\sigma t}, \tag{10}$$

where σ is the angular velocity of the tidal wave; $\overline{W}$, $\overline{W}^*$; $\bar{\xi}$, $\bar{\xi}^*$ are complex amplitudes.

Substituting (10) into the system (6)–(9) and finding the solutions of the ordinary differential equations obtained, with the corresponding boundary conditions, we have

$$\overline{W} = -\frac{ig}{\sigma - \lambda}\frac{\partial\bar{\xi}}{\partial n}\left[1 - \frac{\mathrm{sh}(1-i)\alpha_1 z}{\mathrm{sh}(1-i)\alpha_1 D}\right] + \frac{ig}{\sigma - \lambda}\frac{\partial\bar{\xi}^*}{\partial n}\left[e^{-(1-i)\alpha_1 z} - e^{-(1-i)\alpha_1 D}\frac{\mathrm{sh}(1-i)\alpha_1 z}{\mathrm{sh}(1-i)\alpha_1 D}\right]; \tag{11}$$

$$\overline{W}^* = -\frac{ig}{\sigma + \lambda}\frac{\partial\bar{\xi}}{\partial n}\left[1 - \frac{\mathrm{sh}(1-i)\alpha_2 z}{\mathrm{sh}(1-i)\alpha_2 D}\right] + \frac{ig}{\sigma + \lambda}\frac{\partial\bar{\xi}^*}{\partial n}\left[e^{-(1-i)\alpha_2 z} - e^{-(1-i)\alpha_2 D}\frac{\mathrm{sh}(1-i)\alpha_2 z}{\mathrm{sh}(1-i)\alpha_2 D}\right], \tag{12}$$

where

$$\alpha_1 = \sqrt{(\sigma - \lambda)/2k}; \qquad \alpha_2 = \sqrt{(\sigma + \lambda)/2k}.$$

Using the definition of the functions $\overline{W}$ and $\overline{W}^*$, we find:

$$\bar{u} = -\frac{1}{2}\left\{\frac{ig\,\partial\bar{\xi}/\partial n}{\sigma - \lambda}\left[1 - \frac{\mathrm{sh}(1-i)\alpha_1 z}{\mathrm{sh}(1-i)\alpha_1 D}\right] + \frac{ig\,\partial\bar{\xi}^*/\partial n}{\sigma + \lambda}\left[1 - \frac{\mathrm{sh}(1-i)\alpha_2 z}{\mathrm{sh}(1-i)\alpha_3 D}\right]\right\} +$$

$$+\frac{1}{2}\left\{\frac{ig\,\partial\bar{\xi}/\partial n}{\sigma - \lambda}\left[e^{-(1-i)\alpha_1 z} - e^{-(1-i)\alpha_1 D}\frac{\mathrm{sh}(1-i)\alpha_1 z}{\mathrm{sh}(1-i)\alpha_1 D}\right] + \frac{ig\,\partial\bar{\xi}^*/\partial n}{\sigma + \lambda}\left[e^{-(1-i)\alpha_2 z} - e^{-(1-i)\alpha_2 D}\frac{\mathrm{sh}(1-i)\alpha_2 z}{\mathrm{sh}(1-i)\alpha_2 D}\right]\right\}; \tag{13}$$

$$\bar{v} = -\frac{1}{2}\left\{\frac{g\,\partial\bar{\xi}/\partial n}{\sigma - \lambda}\left[1 - \frac{\mathrm{sh}(1-i)\alpha_1 z}{\mathrm{sh}(1-i)\alpha_1 D}\right] - \frac{g\,\partial\bar{\xi}^*/\partial n}{\sigma + \lambda}\left[1 - \frac{\mathrm{sh}(1-i)\alpha_2 z}{\mathrm{sh}(1-i)\alpha_2 D}\right]\right\} +$$

$$+\frac{1}{2}\left\{\frac{g\,\partial\bar{\xi}/\partial n}{\sigma - \lambda}\left[e^{-(1-i)\alpha_1 z} - e^{-(1-i)\alpha_1 D}\frac{\mathrm{sh}(1-i)\alpha_1 z}{\mathrm{sh}(1-i)\alpha_1 D}\right] - \frac{g\,\partial\bar{\xi}^*/\partial n}{\sigma + \lambda}\left[e^{-(1-i)\alpha_2 z} - e^{-(1-i)\alpha_2 D}\frac{\mathrm{sh}(1-i)\alpha_2 z}{\mathrm{sh}(1-i)\alpha_2 D}\right]\right\}. \tag{14}$$

The relations obtained can be used to calculate the vertical profile of velocity if the magnitude of the turbulence coefficient k, averaged over depth and over the tidal period, is known. To determine the latter, we shall use the equation of the energy balance of turbulence. We integrate it vertically from the surface to the bottom, which makes it possible to eliminate from the equation the term characterizing the diffusive influx of turbulence energy (8). Next we integrate both parts of the turbulence-energy balance equation with respect to time from zero to T, where T is the tidal period. Then, taking into account that from period to period no accumulation or decrease of turbulence energy occurs, the turbulence-energy balance equation is written in the form

Fig. 1. Variation during the tidal period of the components of the current velocity along the parallel (a) and meridian (b). The hours of the tidal period are denoted by Roman numerals.

Figure 1: Fig. 1. Variation during the tidal period of the components of the current velocity along the parallel (a) and meridian (b). The hours of the tidal period are denoted by Roman numerals.

$$\frac{k}{T}\int_0^D\int_0^T\left[\left(\frac{du}{dz}\right)^2+\left(\frac{dv}{dz}\right)^2\right]dt\,dz-c\frac{\bar{b}^{\,2}D}{k}=0, \tag{15}$$

where c is a constant.

The relation between the turbulence energy $\bar{b}$, averaged within the thickness of the sea and over the tidal period, and the turbulence coefficient k will be found using relation (9),

$$k=\bar{l}\sqrt{\bar{b}}, \tag{16}$$

where $\bar{l}$ is the turbulence scale, averaged over height and over the tidal period. To estimate $\bar{l}$ we shall use the representation of the turbulence scale in the form of Zilitinkevich—Laikhtman ([10]), which

for the case of a homogeneous sea reduces to the form:

$$\bar{l}=-\varkappa\Phi'/\Phi'', \tag{17}$$

where

$$\Phi'=\frac{1}{T}\int_0^D\int_0^T\left[\left(\frac{du}{dz}\right)^2+\left(\frac{dv}{dz}\right)^2\right]dt\,dz=\frac{1}{T}\int_0^D\int_0^T\frac{dW}{dz}\frac{dW^*}{dz}\,dt\,dz, \tag{18}$$

primes denote differentiation with respect to z.

Fig. 1. Variation during the tidal period of the components of the current velocity along the parallel (a) and meridian (b). The hours of the tidal period are denoted by Roman numerals.

Substituting into (18) the expressions for W and W^*, and rewriting with the aid of the relation found, equations (15) and (17), we obtain:

$$\bar{l}=-\varkappa\frac{\Pi(D)-\Pi(0)}{\Pi'(D)-\Pi'(0)}; \tag{19}$$

$$k[\Pi(D) - \Pi(0)] = 2cb^2 D/k. \tag{20}$$

Here

$$\Pi' = (du'/dz)^2 + (dv'/dz)^2 + (du''/dz)^2 + (dv''/dz)^2,$$

and u', v', u'', v'' are the velocities of the tidal current at two instants of time separated from one another by a quarter of the tidal period. They are found by separating the real and imaginary parts in (13) and (14).

Eliminating from the system of equations (16), (20), (21) the functions $\bar{l}$ and $\bar{b}$, we obtain an expression for the turbulence coefficient k

$$k = \frac{a}{\sqrt{D}} \frac{[\Pi(D) - \Pi(0)]^{5/2}}{[\Pi'(D) - \Pi'(0)]^2}, \tag{21}$$

where a is a dimensionless parameter. From (21) it is seen that the turbulence coefficient may be regarded as found if the velocity profile of the tidal current under the ice is known.

Thus, by formulas (13), (14), (21) one can simultaneously calculate the vertical profile of the velocity of the tidal current and the value of the turbulence coefficient in a homogeneous sea covered with ice, specifying only the values of the horizontal gradients of the level. In

this coefficient of turbulence is conveniently found by the method of successive approximations.

As an illustration of the application of the proposed model of the structure of a tidal flow, Fig. 1 presents the variation over the tidal period of the vertical distribution of the components of the velocity of the tidal current in a sea covered with ice. In the calculation the following values of the input parameters were adopted: $\sigma = 1.405 \cdot 10^{-4}$ s^{-1}; $\lambda = 10^{-4}$ s^{-1}; $\alpha = 0.5 \cdot 10^3$; $D = 50$ m; $\partial\zeta'/\partial x = -3 \cdot 10^{-7}$; $\partial\zeta'/\partial y = 19 \cdot 10^{-7}$; $\partial\zeta''/\partial x = -15 \cdot 10^{-7}$; $\partial\zeta''/\partial y = -6 \cdot 10^{-7}$. The calculation results agree qualitatively well with Sverdrup' s observations on the Siberian shelf ([1]).

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Note: Figure translations are in progress. See original paper for figures.