ON APPROXIMATION IN THE MEAN OF ANALYTIC FUNCTIONS OF CLASS \(E_p\)

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.537

MATHEMATICS

V. M. KOKILASHVILI

ON APPROXIMATION IN THE MEAN OF ANALYTIC FUNCTIONS OF CLASS E_p

(Presented by Academician N. I. Muskhelishvili on 30 I 1967)

For analytic functions of class E_p, $p > 1$, in a domain with a sufficiently smooth boundary, S. Ya. Al' per ([1]) established an analogue of the well-known Jackson inequality. In the present note, the indicated estimate is sharpened in the sense of order. To this end, for Faber series corresponding to analytic functions of class E_p, $p > 1$, the theorems of Marcinkiewicz ([2]) and Littlewood–Paley ([3]) on multipliers and decomposition of trigonometric Fourier series of functions of class L_p, $1 < p < +\infty$, are generalized. Further, lower and upper estimates are given for deviations in the mean along the contour by linear operators constructed on the basis of the corresponding Faber series.

In the present paper we shall consider analytic functions of class E_p, $p > 1$, in a simply connected domain D with smooth boundary Γ, for which the angle $\theta(s)$ between the tangent and a fixed direction, expressed as a function of the arc length of the curve, has a modulus of continuity satisfying the condition

$$\int_0^\varepsilon \frac{\omega(\delta, \theta)}{\delta}\, d\delta < +\infty, \qquad \varepsilon > 0. \tag{1}$$

$L_p(\Gamma)$ will denote the set of complex-valued measurable functions defined on the curve Γ and satisfying the condition

$$\|f(t)\| = \left\{ \int_\Gamma |f(t)|^p \,|dt| \right\}^{1/p} < +\infty.$$

Suppose that the function $z = \psi(w)$ maps the domain $|w| > 1$ univalently onto the complement of D in such a way that $\psi(\infty) = \infty$ and $\lim_{w\to\infty} \psi(w)/w = 1$.

Under the indicated conditions, for the boundary of the domain D the inequalities ([4])

$$0 < m \le |\psi'(w)| \le M$$

hold for $|w| \geq 1$; M and m are constants depending on the domain. Consequently, if $f(z) \in L_p(\Gamma)$, then the function $f[\psi(w)] \in L_p$ on $|w| = 1$.

We introduce for consideration the quantity

$$\omega_k^{(p)}(\delta, f) = \sup_{|h|\leq\delta} \left\{ \int_0^{2\pi} |\Delta_k^h f_0(\theta)|^p \, d\theta \right\}^{1/p},$$

where

$$f_0(\theta) = f[\psi(e^{i\theta})], \qquad \Delta_k^h f_0(\theta) = \sum_{\nu=1}^{n} (-1)^{k-\nu} \binom{k}{\nu} f_0(\theta + \nu h).$$

$\omega_k^{(p)}(\delta, f)$ is the modulus of smoothness of order k of the function $f(z)$ on Γ.

Next, let $\rho_n^{(p)}(f, \Gamma)$ denote the best approximation in the mean of the p-th degree of the function $f(z) \in E_p$ in the domain D along the contour Γ, i.e.

$$\rho_n^{(p)}(f, \Gamma) = \inf \|f(t) - P_k(t)\|$$

over all polynomials of degree $\leq n$.

Theorem (S. Ya. Al' per). *For every function $f(z) \in E_p$, $p > 1$, in a domain D with boundary satisfying condition (1), the inequality*

$$\rho_n^{(p)}(f, \Gamma) \leq A_1(p, \Gamma) \omega_1^{(p)}(1/n, f) \tag{2}$$

holds.

Another proof of the theorem just formulated, different from the proof of S. A. Al' per, is given in ([5]).

A more precise inequality holds.

Theorem 1. *For every function $f(z) \in E_p$, $p > 1$, the estimate*

$$\frac{1}{n^k} \left\{ \sum_{\nu=1}^{n} \nu^{\beta k - 1} \left[\rho_{\nu-1}^{(p)}(f, \Gamma) \right]^{\beta} \right\}^{1/\beta} \leq A_2(p, k, \Gamma) \omega_k^{(p)} \left(\frac{1}{n}, f \right), \tag{3}$$

where $\beta = \max(2, p)$.

By virtue of the monotonicity of the sequence of best approximations, it is clear that $\rho_n^{(p)}(f, \Gamma)$ is, in order, always no greater than the quantity appearing on the left-hand side of inequality (3). From inequality (3), in particular, it follows that if for $f(z) \in E_p$, $1 < p < +\infty$, for some natural k,

$$\rho_\nu^{(p)}(f,\Gamma) \sim 1/\nu^k,$$

then

$$\omega_k^{(p)}\left(\frac{1}{n}, f\right) \geq \frac{\text{const}}{n^k}(\ln n)^{1/\beta},$$

where $\beta = \max(2, p)$.

Meanwhile, inequality (1) gives only the estimate

$$\omega_k^{(p)}\left(\frac{1}{n}, f\right) \geq \frac{\text{const}}{n^k}.$$

The estimate (3), generally speaking, cannot be improved in order. For $1 < p \leq 2$ this is seen from the following proposition.

Theorem 2. *Let $\mathfrak{M}(a_N)$ denote the class of functions $f(z) \in E_p,\ p \geq 1$, for which*

$$\rho_\mu^{(p)}(f,\Gamma) \sim \alpha_\mu,$$

where $\{\alpha_\mu\}_{\mu=0}^\infty$ is a given monotonically decreasing sequence of numbers tending to zero.

Then

$$\inf_{f\in\mathfrak{M}(a_N)} \omega_k^{(p)}\left(\frac{1}{n}, f\right) \leq \frac{A_3(p,k,\Gamma)}{n^k}\left\{\sum_{\nu=1}^n \nu^{2k-1}\alpha_{\nu-1}^2\right\}^{1/2}.$$

Inequality (3) is unimprovable in order also for other values of p.

In the proof of Theorem 1, an essential role is played by the following assertions, which generalize known results of Marcinkiewicz and Paley–Littlewood on multipliers and decomposition of trigonometric Fourier series.

In what follows, $\{\Phi_n(z)\}_{n=1}^\infty$ will denote the system of Faber polynomials corresponding to the domain D.

Theorem 3. Let in the domain D the function $f(z) \in E_p,\ p > 1$, and let a_n $(n = 0, 1, ...)$ be its Faber coefficients, i.e.

$$a_n = \frac{1}{2\pi i}\int_{|w|=1}\frac{f[\psi(w)]}{w^{n+1}}\,dw.$$

Suppose that $\{\lambda_n\}_{n=0}^{\infty}$ is a sequence of complex numbers satisfying the conditions:

$$|\lambda_n| \leq M, \qquad \sum_{\nu=2^m}^{2^{m+1}} |\lambda_\nu - \lambda_{\nu+1}| \leq M \quad (m = 0, 1, \ldots).$$

Then there exists a function $F(z)$, analytic in the domain D, for which $a_n\lambda_n$ $(n = 0, 1, \ldots)$ are the Faber coefficients, i.e.

$$a_n\lambda_n = \frac{1}{2\pi i}\int_{|w|=1} \frac{F[\psi(w)]}{w^{n+1}}\,dw,$$

and the boundary values along nontangential paths on the contour satisfy the inequality

$$\|F(t)\| \leq A_4(p, \Gamma)M\|f(t)\|. \tag{4}$$

Theorem 4. For every $f(z) \in E_p,\ p > 1$, the inequality

$$A_5(p, \Gamma)\|f(t)\| \leq \left\|\left(\sum_{k=0}^{\infty} |\Delta_k(t)|^2\right)^{1/2}\right\| \leq A_6(p, \Gamma)\|f(t)\|, \tag{5}$$

holds, where

$$\Delta_k(t) = \sum_{\nu=2^{k-1}}^{2^k-1} a_\nu \Phi_\nu(t).$$

Remark. In this theorem $\Delta_k(t)$ may be replaced by

$$\delta_k(t) = \sum_{\nu=n_{k-1}}^{n_k-1} a_\nu \Phi_\nu(t),$$

where $n_{\nu+1}/n_\nu \geq \alpha > 1$ or $n_{\nu+1}/n_\nu \leq \beta < 1$, but then the constants $A_5(p, \Gamma)$ and $A_6(p, \Gamma)$ will depend also on α or β, respectively.

Let $\{\lambda_\nu^{(n)}\}$ $(\nu = 0, 1, \ldots, n;\ n = 1, 2, \ldots;\ \lambda_0^{(n)} = 1,\ \lambda_\nu^{(n)} = 0$ for $\nu > n)$ be a triangular matrix of numbers, and

$$U_n(f; z, \lambda) = \sum_{\nu=0}^{n} \lambda_\nu^{(n)} a_\nu \Phi_\nu(z).$$

For each linear operator $U_n(f; z, \lambda)$, consider the quantity

$$R_n(f; \lambda) = \|f(t) - U_n(f; t, \lambda)\|,$$

which characterizes the deviation of the operator $U_n(f; z, \lambda)$ from the boundary values on the contour in the mean.

Theorem 5. Let

$$\mu_\nu^{(n)} = 1 - \lambda_\nu^{(n)} \quad \text{for} \quad 1 \leq \nu \leq [n^{k/m}],$$

$$\mu_\nu^{(n)} = 0 \quad \text{for} \quad \nu > [n^{k/m}].$$

If

$$\sum_{s=2^\nu}^{2^{\nu+1}-1} \left| \frac{\mu_{s+1}^{(n)}}{(s+1)^m} - \frac{\mu_s^{(n)}}{s^m} \right| \leq \frac{L_1}{n^k} \quad (\nu = 0, 1, \ldots; \ n = 1, 2, \ldots),$$

then for every $f(z) \in E_p,\ p > 1$, the estimate

$$R_n(f; \lambda) \leq A_6(p, \Gamma, \lambda)\, \omega_m^{(p)} \left(\frac{1}{n^{[k/m]}}, f \right). \tag{6}$$

From Theorem 5, estimates of deviations corresponding to concrete summation methods are obtained as corollaries. For example, for the Bernstein–Rogozinski method, the normal Zygmund method, the summation method by Cesàro means of order $\alpha,\ \alpha > 0$, and others.

We give the estimate for Cesàro means of order $\alpha,\ \alpha > 0$.

Corollary. Let

$$\lambda_\nu^{(n)} = \frac{A_{n-\nu}^\alpha}{A_n^\alpha} \quad \text{for } \nu \leq n; \qquad \lambda_\nu^{(n)} = 0 \quad \text{for } \nu > n; \qquad A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}.$$

Then for $f(z) \in E_p,\ 1 < p < +\infty$, the estimate

$$\|\sigma_n^\alpha(f, t) - f(t)\| \leq A_7(p, \Gamma)\, \omega_1^{(p)} \left(\frac{1}{n}, f \right). \tag{7}$$

In the particular case when $\alpha = 0,\ \omega_1^{(p)}(\delta, f) = \delta^\beta,\ 0 < \beta < 1$, estimate (7) was derived in ([1]).

Theorem 6. Let $0 \leq \lambda_s^{(n)} \leq 1$ $(s = 1, 2, \ldots)$.

If

$$\sum_{s=2^\nu}^{2^{\nu+1}-1} \left| \frac{(s+1)^k}{\mu_{s+1}^{(n)}} - \frac{s^k}{\mu_s^{(n)}} \right| \le L_2 n^k \quad (\nu = 0, 1, \dots;\ n = 1, 2, \dots),$$

then for any $f(z) \in E_p,\ p > 1$, the estimate

$$\frac{1}{n^k} \left\{ \sum_{\nu=1}^{n} \nu^{\beta k - 1} \left[\rho_\nu^{(p)}(f, \Gamma) \right]^\beta \right\} \le A_8(p, \Gamma) \left\| f(t) - U_n(f; t, \lambda) \right\|, \tag{8}$$

where $\beta = \max(2, p)$.

Finally, let us note that the main theorems of the constructive theory of functions for $L_p(0, 2\pi),\ 1 < p < +\infty$, have been sharpened in order by M. F. Timan ([6]).

Estimates analogous to (6) and (8), for periodic functions of class $L_p,\ 1 < p < +\infty$, are contained in works ([7,8]).

Tbilisi Mathematical Institute
named after A. M. Razmadze
Academy of Sciences of the Georgian SSR

Received
26 I 1967

REFERENCES

1. S. Ya. Al' per, *Investigations on Contemporary Problems in the Theory of Functions of a Complex Variable*, 1960, pp. 273–286.

2. J. Marcinkiewicz, *Studia Math.*, **8**, 78 (1939).

3. J. Littlewood, R. Paley, *Proc. London Math. Soc.*, **42**, 52 (1936).

4. S. Warschawski, *Math. Zs.*, **35**, 321 (1932).

5. M. I. Andrashko, *Problems of Mathematical Physics and the Theory of Functions*, No. 1, Kiev, 1964, p. 3.

6. M. F. Timan, Abstract of doctoral dissertation, Tbilisi, 1962.

7. V. M. Kokilashvili, *Communications of the Academy of Sciences of the Georgian SSR*, **43**, No. 2, 257 (1966).

8. V. V. Zhuk, *Doklady AN*, **169**, No. 3, 515 (1966).

Note: Figure translations are in progress. See original paper for figures.