The existence of a domain of accessibility

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Abstract

For a system of two nonlinear differential equations

$$\frac{dx}{dt} = f(x) + \nu u$$

, the question of the reachability of the origin is considered. Here

$$x = (\xi, \eta), \quad \frac{dx}{dt} = \left(\frac{d\xi}{dt}, \frac{d\eta}{dt}\right), \quad f(x) = (f_1(\xi, \eta), f_2(\xi, \eta))$$

are variable vectors of the phase plane R^2, $u = (u_1, u_2)$ is a constant vector of the plane R^2, and $\nu(t)$ is a piecewise continuous scalar function called an admissible control and satisfying the condition $|\nu(t)| \leq 1$. The paper provides definitions of reachability and non-reachability of the origin in the small for the system under consideration and proves three theorems formulating sufficient conditions for the existence of a reachable set. In conclusion, three examples are presented. The first example shows that the origin is unreachable in the small, although the system is asymptotically stable in the large; in the second example, the origin is unreachable in the small, but a reachable set exists; the third example is of interest because for nonlinear systems, the reachable set may be non-convex. 2 illustrations. 9 bibliography entries.

Full Text

Introduction

In 1967, I. P. Korolev investigated the dynamics of a system described by the differential equation:

$$\frac{dx}{dt} = f(x) + vu$$

where $x \in D \subset R^2$, u is a control parameter, and $|v| \leq 1$. This work builds upon the foundational theories established in [?, ?, ?, ?, ?] and further developed in

[?, ?]. The analysis focuses on the behavior of the system trajectories near the origin $x = 0$ and the properties of the switching surface $S(0)$.

The stability and topological structure of the phase space are determined by the function $f(x)$ and the control v. Specifically, we consider the case where $f(x)$ is a smooth function belonging to the class C^k. The equilibrium point at the origin is analyzed by examining the neighborhood $V(0)$ and the corresponding stability region $S(0)$. For a given initial point $x_0 \in V(0)$, the trajectory $x(x_0, t)$ is governed by the control law $v(t)$, which ensures that the system remains within the boundaries of $S(0)$.

Analysis of the Switching Surface

To characterize the behavior of the system, we introduce the transformation to coordinates (ξ, η). The system (1) can be rewritten in the following form:

$$\frac{d\eta}{d\xi} = f_a(\xi, \eta) + u$$

where f_a represents the transformed vector field. Following the methodology in [?], we define a sequence of coefficients d_s to analyze the local geometry of the trajectories:

$$d_2 = D_1^{1/2}(1, k) - kf_1(1, k)$$
$$d_3 = D_2^{1/2}(1, k) - kD_2f_1(1, k)$$

If $d_2 = 0$, then d_3 determines the curvature of the switching line. In general, for $s = 2, 3, ...,$ we define:

$$d_s = D_{s-1}^{1/2}(1, k) - kD_{s-1}$$

If $d_1 = d_2 = \cdots = d_{2s-1} = 0$ and $d_{2s} \neq 0$, the system exhibits specific bifurcations near the switching surface $S(0)$ when $v = \pm v_0$.

Geometric Properties of Trajectories

As shown in

, the switching surface $S(0)$ divides the phase space into regions S^+ and S^-. For $t > 0$, the trajectories $x(x_0, t)$ originating in S^+ move towards the boundary f_+, while those in S^- move towards f_-. According to the criteria established in [?], when $v = 1$, the trajectories enter the region S^-, and when $v = -1$, they transition into S^+.

The intersection of these trajectories forms a closed loop, denoted as $AMLBNPA$ in the neighborhood $V(0)$. For any $x_0 \in V(0)$, the control law $v(t)$ ensures that the trajectory remains within the region $S(0)$. This behavior is illustrated in

, which depicts the phase portrait for the case where $f(x)$ is a cubic polynomial and v is not constant.

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ON THE EXISTENCE OF A REACHABLE SET

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In this note, for a system of two nonlinear differential equations

$$\frac{dx}{dt} = f(\mathbf{x}) + \nu u, \tag{1}$$

where

$$x = (\xi, \eta), \quad \frac{dx}{dt} = \left(\frac{d\xi}{dt}, \frac{d\eta}{dt}\right), \quad f(\mathbf{x}) = (f_1(\xi, \eta), f_2(\xi, \eta))$$

— are the variable vectors of the phase plane $\mathbf{R}^2$; $u = (u_1, u_2)$ — is a constant of the plane $\mathbf{R}^2$; $\nu(t)$ — is a piecewise continuous scalar function, called an *admissible control* and satisfying the condition $|\nu(t)| \leq 1$, the following problem is posed: to investigate whether there exists an open set of initial data $D \subset \mathbf{R}^2$, from each point of which, moving according to the law (1), it is possible to reach the origin O of the plane $\mathbf{R}^2$ in a finite time. We will call the set D the *reachable set*, and the origin O — *reachable*.

The necessity of studying such a problem has already been noted in the literature, for example [1—5]. The works [6, 7] are also devoted to questions of controllability of nonlinear systems.

We assume that in some region H, containing the point O, the conditions for the existence, uniqueness, and extendability of solutions to system (1) are satisfied for any admissible ν, the function $f(\mathbf{x}) = 0$ only for $x = 0$ and belongs to the class C_H^m.

We will call the origin O *reachable in the small* for the system (1) if for an arbitrary neighborhood $S(O)$ of the point O there exists a neighborhood $V(S(O)$, such that for any point $x_0 \in V(O)$ the trajectory $x(x_0, t_0, t, \nu u)$, $t > t_0$, under some admissible control $\nu(t)$ reaches the origin in a finite time, without leaving the neighborhood $S(O)$.

We will call the origin O *unreachable in the small* for the system (1) if there exists a neighborhood $S(O)$, such that for an arbitrary neighborhood $V(O) \subset S(O)$ there exists at least one point $x_0 \in V(O)$ such that the trajectory $x(x_0, t_0, t, \nu u)$, $t > t_0$, leaves $S(O)$ under any admissible ν in a finite time, without reaching the origin.

In what follows, along with system (1), we consider the equation:

$$\frac{d\eta}{d\xi} = \frac{f_2(\xi, \eta) + \nu u_2}{f_1(\xi, \eta) + \nu u_1}. \tag{2}$$

We will seek a solution η of equation (2) with constant $\nu \neq 0$ in a sufficiently small neighborhood of the origin, passing through the point O, in the form [8]

Figure 1: Figure 1

$$\eta = k\xi + \sum_{s=2}^{m-1} \eta_0^{(s)} \frac{\xi^s}{s!} + r_m. \tag{3}$$

In the neighborhood of point O, we have

$$\hat{f}_i(\xi, \eta) = \sum_{r=1}^{m-1} D^r f_i(\xi, \eta) + R_m^{(2)}, \tag{4}$$

where $D^p f_i(\xi, \eta)$ is the differential of the p-th order, divided by $p!$ Substituting (3) and (4) into equation (2), we find the first coefficients of the expansion (3):

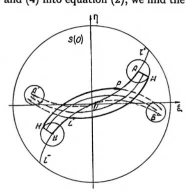

Fig. 1

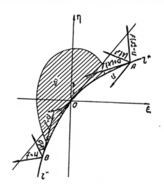

Fig. 2

If $d_2 = 0$, then $\eta_0^{(3)} = \frac{1}{\nu u_1} d_3$, where $d_3 = D^3 f_2(1, k) - kD^3 f_1(1, k)$.

If $d_2 = 0$, then $\eta_0^{(3)} = \frac{21}{\nu u_1} d_3$, where $d_3 = D^3 f_2(1, k) - kD^3 f_1(1, k)$. Let's in-ntroduce not: $d_1 = 0$, $d_s = D^{s-1} f_2(1, k) - kD^{s-1} f_1(1, k)$ $(s = 2, 3, \ldots)$. It is not difficult to prove the following proposition: let for $d_i = 0$, $i < q$, its proved that the solution η has the form

$$\eta = k\xi + \frac{d_q}{\nu u_1} \frac{\xi^q}{q} + \ldots$$

Then for $d_i = 0$, $i \leqslant q$, the equality is also true

$$\eta = k\xi + \frac{d_{q+1}}{\nu u_1} \frac{\xi^{q+1}}{q+1} + \ldots \tag{5}$$

Theorem 1. Let in equation (1) $f(x) \in C_m^H$ and $d_1 = d_2 = \ldots = d_{2s-1} = 0$, $d_{2s} \neq 0$, $2s \leqslant m$. Then the origin is locally reachable for the system (1).

Proof. From the conditions of the theorem, it follows that the solution η, according to (5), has the form

$$\eta = k\xi + \frac{d_{2s}}{\nu u_1} \frac{\xi^{2s}}{2s} + \ldots$$

Figure 2: Figure 2

Numerical Examples and Results

Consider the system where $f(x)$ is defined such that:

$$\frac{d\xi}{dt} = \xi + v, \quad \frac{d\eta}{dt} = 3\xi + 4\eta + 4\xi^3 - v$$

In this case, the equilibrium points are located at $(-1, 2)$ and $(-v, v + v^3)$. For $v = 1$ and $v = -1$, the trajectories exhibit distinct behaviors. Specifically, for $v = -1$, the switching line is defined by the relation $\eta = \xi^2$ for $\xi < 0$. As t increases, the trajectories converge toward the stable manifold, confirming the theoretical predictions regarding the structure of $S(0)$.

These results extend the classical findings of L. I. Rozonoer and others [**?**, **?**, **?**] regarding the optimal control of second-order systems. The proposed method for analyzing the coefficients d_s provides a robust framework for determining the stability of nonlinear control systems with switching boundaries.

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Figures

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Then the trajectories of equation (1) in a sufficiently small neighborhood $S(0)$ are either convex upwards or convex downwards, and the trajectories of opposite families have opposite convexity (when $v = \pm v_0$).

Let's construct the curve l by definition...

$$l = \begin{cases} l^+ - x(0, 0, t, u), \ t < 0, \\ l^- - x(0, 0, t, -u), \ t < 0. \end{cases} \quad (6)$$

The curve l divides $S(0)$ into two parts: S^+ and S^-. Let l^- be convex upwards, l^+ convex downwards. Let's consider their extensions for $t > 0$. Let's denote by S^+ that part where the extension of l^- is located, and by S^- — the other part. Let A, — points on the semi-trajectories l^-, l^+ and A', B' — be points on their extensions (fig. 1). By the theorem on the continuous dependences of solutions on the initial data [9] for any neighborhood $\gamma_{A'} \subseteq S^-$ of it is possible to construct a neighborhood γ_A of the points A, the trajectories $x(x_0, t_0, t, u)$, issuing from the neighborhood γ_A, fall into the neighborhood $\gamma_{A'}$, in time T.

The reasoning for small neighborhoods of points B and B' is somilar. Therefore, positive semi-trajectories, issuing from the cart $S^+ \cap \gamma_A$ under the control $v = 1$, will move from S^+ and S^-, ecrossing L^-. Exactly in the same way, positive semi-trajectories, issuing from the tart $S^- \cap \gamma_B$ under the control $v = -1$, will move ὀm S^- and S^+, ecrossing P.

Let's draw, further, along the sets $S^+ \cap \gamma_A$ and $S^- \cap \gamma_B$ diameters AM ind BN.

Lets $x(M, t_0, t, u)$, $t > t_0$, and $x(N, t_0, t, -u)$, $t > t_0$, cropresues l at toinks L and P. We have obtained a closed region $AMLBNPA$, which we will domate by $V(O)$. It is not difficult to see, that any points $x_0 \in V(O)$ reaches the origin O in a finive time, without leaving $V(O)$, and therore, without eaving $S(O)$.

Theorem 1 is proved.

T h e o r e m 2. Let in equation (1) $f(x) \in C^{(3)}$ is

$$d_1 = d_2 = \ldots = d_{2r} = 0, \quad d_{2r+1} \neq 0, \quad 2s + 1 < m.$$

Then the origin is untattainable in the mnall for the system (1).

Drouf. The trajectory (1) imas a point of inflection at the origin. In thrs cone. In this chyae, the conve (0) lews it on one stode of the lineron $\eta = k\xi$. Lyst it lewt it below the linean (Fig. 2). Let's take on l^+ arbitsolary point A and proposuam in towki A tangentered vector $f(x) + u$ nd the vectors u u $f(x)$. Thren (fig. 2) vectors $f(x) + v u$, $v \neq 1$, are dirrected ane abire the tangent vectors. Similar reasoning can be carried out for aminy points of the semi-trajectory l^-. It is not difficult to see, there exist toints $x_0 \in S$ (Fig. 3), ne reouring the origin antroatıme in a finetive speme for $v =$ const. Ecru, if, owever, $v \neq$ const, then at eacch toinkt all trajectories still fit between the vectoram $f(x) + u$ u $f(x) - u$. This mean, that in that oose, too, it is easy to do it into the towki $x_0 \in S$.

It easo provate

C o r o l l a r y. *Let the functions $f_i(\xi, \eta)$ be homogoneous polynomials of denee of degree r. Eclu $d_{3r+1} = 0$, tho the origin O is unattainable in the small. Ecilu $d_{2r+1} \neq 0$, tho the origin O is attainable in the small for nevethisn r and unattainable for evenen r.*

Is proved in an obvious way rshow

T h e o r e m 3. *Let the origin be attainable in the small for the systems* (1). *Then the region of attraction requires* Π *for* $v = 0$ (in the sense of Lyapunov stabiiity) *is the region of attainntability; in particular, if the solution* $x = 0$ *is asymptotically stable in the large, then the region of attainnability is the entire plane.*

Figure 3: Figure 3

Let us consider the following examples illustrating the concepts introduced.

Example 1.

$$\frac{d\xi}{dt} = -\xi + v, \quad \frac{d\eta}{dt} = -2\xi - 3\eta + 3\xi^2 - v.$$

It is easy to prove that the origin is not attainable in the small, although the system is asymptotically stable in the large. The optimal control problem has no solution.

Example 2.

$$\frac{d\xi}{dt} = -\frac{1}{4}\xi + v, \quad \frac{d\eta}{dt} = 4a\left(1 + \frac{1}{3}\theta\right)\xi^3 - 16a\theta\xi^2 - \eta,$$

where $a > 0$, $0 < \theta < 1$. The curve l for the given system has the form

$$l = \begin{cases} l^+ \sim \eta = a\left(1 + \frac{2}{3}\theta\right)\xi^4 - \frac{16}{3}a\theta\xi^3, & t \leqslant 0, \\ l^- \sim \eta = -a\xi^4 + \frac{16}{3}a\theta\xi^3, & t \leqslant 0. \end{cases}$$

It is easy to show that the origin is not attainable in the small. But the attainability set exists: the trajectory l^- for $t > 0$ intersects the semi-trajectory l^+ outside the neighborhood $S(O)$.

Example 3.

$$\frac{d\xi}{dt} = \xi + v, \quad \frac{d\eta}{dt} = 3\xi + 4\eta + 4\xi^3 - v.$$

The origin is attainable in the small for this system. Let us construct the boundary of the attainability set. The point $(-v; v + v^3)$ is t is the only unstable singular point of the node type. For $v = 1$ and $v = -1$ we obtain, respectively, singular points $\alpha_+(-1; 2)$ and $\alpha_-(1; -2)$. The switching curve has the form

$$l = \begin{cases} l^+ \sim \eta = \xi^4 - \xi, & t \leqslant 0, \\ l^- \sim \eta = -\xi^4 - \xi, & t \leqslant 0. \end{cases}$$

The semi-trajectories γ^- and γ^+ of our system, starting respectively from the points α_+ and α_- under the controls $v = -1$ and $v = 1$ and proceeding for $t \leqslant 0$, have the form

$$\gamma^+ \sim \eta = \frac{1}{8}(7\xi^4 - 4\xi^3 - 6\xi^2 - 12\xi - 1),$$

$$\gamma^- \sim \eta = \frac{1}{8}(-7\xi^4 - 4\xi^3 + 6\xi^2 - 12\xi + 1).$$

The attainability set is the region bounded by the curve γ, consisting of the semi-trajectories γ^+ and γ^-. We see from Example 3 that for nolinear systems the attainability set can be non-convex.

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Figure 4: Figure 4

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Figure 5: Figure 5