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Date: 1967-01-01T00:00:00+00:00

Abstract

The equation $\ddot{x} = f(x) - R(x, \dot{x})$ with a discontinuous right-hand side, which can be interpreted as a generalized pendulum equation, is considered. This work is devoted to the issues of the existence and relative positioning of global limit cycles. Estimates of the distance between two global limit cycles are obtained.

Illustrations: 2. Bibliography: 5.

Full Text

Preamble

DIFFERENTIAL EQUATIONS DECEMBER 1967, VOLUME III, No. 12

Consider the equation $\ddot{x} + R(x, \dot{x}) = f(x)$, which is equivalent to the system:

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -R(x, y) + f(x)\end{aligned}$$

where $f(x)$ is defined such that: 1) The functions $R(x, y)$ and $f(x)$ are continuous for all real values and are continuously differentiable in the neighborhood of the singular points of equation (1). To avoid critical cases, it is also assumed that the derivatives of the functions $f(x)$ and $R(x, y)$ do not vanish at the zeros of these functions. 2) The functions $f(x)$ and $R(x, \dot{x})$ are periodic with period 2π . The function $f(x)$ has a finite number of zeros within one period:

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The function has zeros within its period: $n + m\tau$, where n and m are any finite integers. 4) $\Delta(x)$ changes sign—the first when passing through the point x , and the second when passing through...

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- 5) $\int_0^{2\pi} U(x)dx > 0$. 6) $R(x, x)$ is increasing with respect to x . 7) $R(x, 0) = 0$ for any x . 8) $\lim R(x, x) > f_1(x)$, $\lim R(x, x) < f_2(x)$.

Equation (1) can be interpreted as describing the motion of a material point along a closed curve under the influence of forces depending on position and velocity—specifically, under the action of a driving (pushing) force. Studies of this equation under similar constraints have been conducted in works [?, ?, ?, ?] and others. The presence of a driving force determines the following relative arrangement of the zeros of the functions $f_1(x)$ and $f_2(x)$:

$$x_0 < \eta_0 < \eta_1 < x_1 < x_2 < \dots < x_{2k} <$$

$< \eta_{2k} < \eta_{2k+1} < \dots < \eta_{2m+1} < x_{2m+1} < \dots < x_{2n-1}$. In particular, when $n = m$, we have:

$$x_0 < \eta_0 < \eta_1 < x_1 < x_2 < \dots < x_{2k-1} < x_{2k} <$$

$$< \eta_{2k} < \eta_{2k+1} < \dots < \eta_{2m-1} < x_{2n-1}.$$

For simplicity of presentation, we shall henceforth assume that $n = m$. The possibility of the existence of limit cycles in the phase plane in this case, which encompass one or several instability segments of the type $[\eta_{2i+1}, x_{2i+1}]$, has been demonstrated in [?, ?]. Let us consider in more detail the questions regarding the existence and relative positioning of “all-encompassing” limit cycles. We define an all-encompassing limit cycle of equation (1) in domain D as a limit cycle that encloses all instability segments of the type $[\eta_{2i+1}, x_{2i+1}]$ lying within that domain.

Theorem. If in domain D , for which one of the sufficient conditions for the existence of a limit cycle is satisfied, the instability segments $[\eta_{2i+1}, x_{2i+1}]$ are located inside regions D_1 and D_2 bounded by separatrices and segments of the Ox axis, and phase trajectories can only exit the regions D_1 and D_2 , then there exists at least one all-encompassing limit cycle in domain D .

Proof. By the assumption that the regions D_1 and D_2 contain all unstable segments $[\eta_{2i+1}, x_{2i+1}]$, these regions will not contain only the single unstable segment of type $[\eta_{2i}, x_{2i}]$ lying in domain D . That is, region D_1 will be bounded by separatrices adjacent to point $(x_0, 0)$, and region D_2 will be bounded by separatrices adjacent to point $(x_{2n}, 0)$

. Phase trajectories only exit regions D_1 and D_2 , while they only enter domain D . Thus, phase trajectories only enter the region $D \setminus (D_1 \cup D_2)$. Since there are no stable singular points within this region, by the Bendixson theorem, there must exist a stable limit cycle in this region. This cycle will be all-encompassing, as it cannot enclose only region D_1 or D_2 due to the fact that it cannot cross an instability segment.

Remark 1. In each of the regions D_1 and D_2 , either no all-encompassing limit cycle exists at all, or there is an even number of them.

ON THE EXISTENCE OF A REACHABILITY REGION

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where

$$x = (\xi, \eta), \quad \frac{dx}{dt} = \left(\frac{d\xi}{dt}, \frac{d\eta}{dt} \right), \quad f(x) = (f_1(\xi, \eta), f_2(\xi, \eta)) \quad (1)$$

In this note for a system of two nonlinear differential equations — variable vectors of the phase plane R^2 ; $u = (u_1, u_2)$ — a constant vector of the plane R^2 ; $v(t)$ — a piecewise continuous scalar function, called an *admissible control*, and satisfying the condition $|v(t)| \leq 1$, at first, the following problem is posed: to investigate whether there exists an open region of initial data $D \subset R^2$, from each point of which, moving according to law (1), it is possible to reach the origin, origin O of the plane R^2 in finite time. The region D will be called the *reachability region*, and the origin O — *reachable*.

The necessity of studying such a problem has already been noted in the literature, for example [1–5]. The works [6, 7] are also devoted to questions of controllability of nonlinear systems.

We assume that in some region H , containing the point O , the conditions of existence, uniqueness and continuability of solutions of system (1) are satisfied for any admissible v , the function $f(x) = 0$ only at $x = 0$ and belongs to class C_{ij}^∞ .

The origin O is called *reachable in the small* for system (1), if for an arbitrary neighborhood $S(O)$ of point O there is found a neighborhood $V(O) \subset S(O)$, such that for any point $x_0 \in V(O)$ are trajectory: $S(O)$, such that for any point $x_0 \in V(O)$ the commensurate $x_0 \in V(O)$ the trajectory $x(x_0, t_0, t, v, u)$, $t > t_0$, under some admissible control $v(t)$ reaches the origin in finite time, without leaving the neighborhood $S(O)$.

The origin O is called *unreachable in the small* for system (1), if there is found a neighborhood $S(O)$, that for an arbitrary neighborhood $V(O) \subset S(O)$ there is found at least one point $x_0 \in V(O)$ such that the trajectory $x(x_0, t_0, t, v, u)$, $t > t_0$, the trajectory $x(x_0, t_0, t, v, u)$, $t > t_0$, leaves $S(O)$ contrat under any admissible v in finite time, not reaching the origin.

Further, along with system (1) we consider the equation:

$$\frac{d\eta}{d\xi} = \frac{f_2(\xi, \eta) + v u_2}{f_1(\xi, \eta) + v u_1}. \quad (2)$$

We will seek a solution η of equation (2) with constant $v \neq 0$ in a sufficiently small neighborhood of the origin, passing through point O , in the form [8]

Figure 1: Figure 1

Remark 2. If integral curves only exit the regions D_1 and D_2 , then an all-encompassing limit cycle in domain D is either absent or there is an even number of them.

Remark 3. If the region D is bounded by separatrices for which the following equality holds:

$$x\{R\% k\} = x\{S\ 2k\},$$

...or by separatrices for which the equality holds:

$$X(R2s) = * x\{S' \ 2s\},$$

Consequently, we observe the degeneracy of the separatrices themselves into an all-encompassing limit cycle. Let η_1, η_2 denote the abscissas of the intersection points of the separatrices with the x -axis that are closest to the origin. Suppose that $|f_1(x) - f(x)| < \delta$ within the domain G .

Theorem 2

In a domain D containing the instability intervals $[\eta_3, \eta_4]$, there exist all-encompassing limit cycles. If $\frac{dR(x,y)}{dx} > m > 0$, then the distance between any pair of these cycles exceeds a specific magnitude. Here, η_3 and η_4 represent the abscissas of the left and right intersection points of the boundary of domain D with the x -axis, respectively.

The validity of this theorem follows directly from Theorem 1 [?]. Indeed, by considering the interval $[\omega_2, \omega_1]$ as an interval known not to contain any points lying on the all-encompassing limit cycles, we satisfy the conditions of the aforementioned theorem.

Theorem 3

If n instability intervals $[\eta_3, \eta_4]$ of the system are located within a certain number of limit cycles V_k , and the cycles T_k are, in turn, enclosed by a limit cycle V , then the cycle V cannot be separated from the cycles T_k by a distance greater than...

$$\bullet \ ' = (k+1)dp \ 0$$

To prove this, let us assume the existence of a cycle Γ' distinct from Γ . For a closed contour consisting of limit cycles (Fig. 2), we have, by virtue of system (2):

$$J = \oint [R(x, y) + f(x)]dx + ydy = \oint [R(x, y) + f(x)]dx \quad (4)$$

We now represent the integral in expression (4) over the closed contour by isolating the boundaries of the ϵ -neighborhoods of the segments of the x -axis located between cycles Γ' and Γ as separate contours. This approach is necessary

due to the non-uniqueness of the function $f(x)$ on the x -axis; we consider ϵ to be arbitrarily small. Applying Green's theorem, we can write:

$$\oint f(x)dx + \oint f(x)dx + \dots + \oint f(x)dx$$

Obviously, as $\epsilon \rightarrow 0$, we have:

$$\iint_G \frac{\partial R(x, y)}{\partial y} dx dy + \oint [f(x) - h(x)] dx + \oint [f(x) - h(x)] dx$$

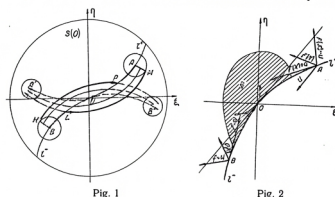
Furthermore, we obtain $(k + 1)d\rho$, where G denotes the region lying between Γ' and Γ , and S denotes the area of this region.

$$\eta = k\xi + \sum_{s=2}^{m-1} \eta_0^{(s)} \frac{\xi^s}{s!} + r_m. \quad (3)$$

In okignocyt of tonkt O we have

$$f_i(\xi, \eta) = \sum_{r=1}^{m-1} D^r f_i(\xi, \eta) + R_m^{(i)}. \quad (4)$$

rde $D^p f_i(\xi, \eta)$ — the differential p -th order, divided by $p!$ Substituting (3) and (4) in γ equation (2), we find the first coefficients of the expansion (3):



$$k = \frac{u_2}{u_1}, \eta_0^{(2)} = \frac{1}{v u_1} d_2, \text{ where } d_2 = D^2 f_2(1, k) - k D^2 f_1(1, k).$$

Если $d_2 = 0$, то $\eta_0^{(2)} = \frac{2!}{v u_1} d_3$, there $d_3 = D^3 f_2(1, k) - k D^3 f_1(1, k)$. Bredum oboznavemue: $d_i = 0, d_i = D^{i+1} f_2(1, k) - k D^{i+1} f_1(1, k)$ ($i = 2, 3, \dots$). It is table to doravite takon proposition: nyctz that $d_i = 0, i < q$, dokassano, who peuenue η umeet buq

$$\eta = k\xi + \frac{d_q}{v u_1} \frac{\xi^q}{q} + \dots$$

Torda for $d_i = 0, i < q$, the paventy cawes true

$$\eta = k\xi + \frac{d_{q+1}}{v u_1} \frac{\xi^{q+1}}{q+1} + \dots \quad (5)$$

Thopema 1. Lyet in ypadnenion (1) $f(x) \in C_m^q, d_1 = d_2 = \dots = d_{m-1} = 0, d_m \neq 0, 2s \leq m$. Toeda navalo koordynant dostuzhimo in mall das custemer (1). Dokassatenyctso. His ycloition teorem cledyes, the peuenue η , coordino (5), umeet buq

$$\eta = k\xi + \frac{d_{2s}}{v u_1} \frac{\xi^{2s}}{2s} + \dots$$

Figure 2: Figure 2

It is not difficult to see that:

$$S > 2\rho(x_{2n-1} - x_1)$$

Moreover, we replace the segment $[a_{i-1}, a_i]$ of the axis between two adjacent limit cycles Γ_{i-1} and Γ_i with a larger axis segment between the end of the rightmost

instability segment lying inside Γ_{i-1} and the beginning of the leftmost instability segment lying inside Γ_i . We then take ρ to be the length of the largest of these segments. Then, according to the condition of the theorem, we have:

$$J < -2 \text{ m p } (\text{ x } 2 \text{ r t } _ ! \text{ -rix }) + (\text{k} + 1) \text{rfp } 0 < 0,$$

This leads us to a contradiction. Thus, we have demonstrated that the distance ρ between any two encompassing limit cycles does not exceed the values determined by Theorem 2 and Theorem 3. However, when determining ρ , the value ρ^* generally cannot be calculated directly. Therefore, instead of using the exact separatrices that bound the region D , one may employ their respective upper or lower approximations [?]. Furthermore, the abscissas of the intersection points between these approximating curves and the axis may be taken as η^* . In this manner, the value of ρ can be effectively computed.

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Received by the editorial office on June 5, 1967. A. M. Gorky Ural State University

Figures

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Then the trajectories of equation (1) in a sufficiently small neighborhood $S(O)$ are other convex v upwards, or convert downwards, and the trajectories of opposite families have opposite convexity (when $v = \pm v_0$).

Let us construct the curve l by definition

$$l = \begin{cases} l^+ \sim x(O, 0, t, u), & t \leq 0, \\ l^- \sim x(O, 0, t, -u), & t \leq 0. \end{cases} \quad (6)$$

Curve l divides $S(O)$ into two parts: S^+ and S^- . Let l^- be convex upwards. l^+ be convex downwards. Consider their continuation $t > 0$. Let us denote by S^- the part where the continuation P is located, a S^+ — the other part.

Let A, B — points on the semi-trajectories P, l^- and A', B' — points on their continuations (Fig. 1). By the theorem on continuous dependence of solutions on initial data [9], for any neighborhood $\gamma_A \subset S^-$ of point A , to construct a neighborhood γ_A of point A such that the trajectories $x(x_0, t_0, t, u)$, issuing from the neighborhood γ_A , fall into the neighborhood γ_A , in time T .

Arguments for some neighborhoods of points B and B' are analogous. Therefore, positive semi-trajectories issuing from the part $S^+ \cap \gamma_A$ and control $v = 1$, will pass from S^+ to S^- , crossing l^- . Exactly in the same way, positive semi-trajectories issuing from the part $S^+ \cap \gamma_B$ and control $v = -1$, will pass from S^+ to S^- , crossing l^+ .

Let us draw, further, on the sets $S^+ \cap \gamma_A$ and $S^- \cap \gamma_B$ the diameters AM and BN .

Let $x(M, t_0, t, u), t > t_0$, and $x(N, t_0, t, -u), t > t_0$, crosses l^+ at points L and P . We will obtain a closed region $AMLEPA$, which we will denote by $V(O)$. It is not difficult to see that any point $x_0 \in V(O)$ reaches the origin O in finite time, without leaving $V(O)$, and thus, without leaving $S(O)$.

Theorem 1 is proved.

Theorem 2. Let in equation (1) $f(x) \in C^H_H$ and

$$d_1 = d_2 = \dots = d_{2s} = 0, \quad d_{2s+1} \neq 0, \quad 2s + 1 \leq m.$$

Then the origin is unreachable in the small for system (1).

Proof. Trajectory (3) has an inflection point at the origin. In this case, curve (6) lies on one side of the line $\eta = k\xi$. Let it lie below the line (Fig. 2). Let us take on l^+ an arbitrary point A and draw from point A a tangent vector $f(x) + u$ and vectors u and $f(x)$. Let a vectors u $f(x)$. Then (Fig. 2) vectors $f(x) + vu, v \neq 1$, are directed above the tangent vector. Analogous arguments can be carried out for any point of semi-trajectory l^- . It is easy to see that there exist points $x_0 \in S$ (Fig. 2), not reaching the origin in finite time for $v = \text{const}$. If $v \neq \text{const}$, then at each point also all trajectories are placed between vectors $f(x) + u$ and $f(x) - u$. Therefore, also in this case, it is impossible to reach the origin from points $x_0 \in S$.

It is easy to prove

Corollary. Let functions $f_s(\xi, \eta)$ be homogeneous polynomials of degree r . If $d_{r+1} = 0$, then the origin O is unreachable in the small. If $d_{r+1} \neq 0$, then the origin O is reachable in the small for odd r and unreachable for even r .

It is proved in an obvious way

Theorem 3. Let the origin be reachable in the small for system (1). Then the region of attraction Π for $v = 0$ (in the sense of stability according to Lyapunov) is the region of reachability; in particular, if the solution $x = 0$ is asymptotically stable in the whole, then the region of reachability is the whole plane.

Figure 3: Figure 3

Let us consider the following examples, illustrating the introduced concepts.
 Example 1.

$$\frac{d\xi}{dt} = -\xi + v, \quad \frac{d\eta}{dt} = -2\xi - 3\eta + 3\xi^2 - v.$$

It is not difficult to prove that the origin is unreachable in the small, although the system is asymptotically stable in the large. The optimal control problem has no solution.

Example 2.

$$\frac{d\xi}{dt} = -\frac{1}{4}\xi + v, \quad \frac{d\eta}{dt} = 4a \left(1 + \frac{1}{3}\theta\right)\xi^3 - 16a\theta\xi^3 - \eta,$$

where $a > 0, 0 < \theta < 1$. The curve l for this system has the form

$$l = \begin{cases} l^- \sim \eta = a \left(1 + \frac{\theta}{3}\right)\xi^3 - \frac{16}{3}a\theta\xi^3, & t \leq 0, \\ l^- \sim \eta = -a\xi^1 + \frac{16}{3}a\theta\xi^3, & t \leq 0. \end{cases}$$

It is easy to show that the origin is unreachable in the small. But the reachable set exists: the trajectory l^- for $t > 0$ intersects the semi-trajectory l^+ outside the neighborhood $S(O)$.

Example 3.

$$\frac{d\xi}{dt} = \xi + v, \quad \frac{d\eta}{dt} = 3\xi + 4\eta + 4\xi^3 - v.$$

The origin is reachable in the small for this system. Let us construct the boundary of the reachable set. The point $(-v; v + v^3)$ is the unique unstable singular point of the node type. For $v = 1$ and $v = -1$ we obtain, respectively, the singular points $\alpha_+(-1; 2)$ and $\alpha_-(1; -2)$. The switching curve has the form

$$l = \begin{cases} l^+ \sim \eta = \xi^4 - \xi, & t \leq 0, \\ l^- \sim \eta = -\xi^1 - \xi, & t \leq 0. \end{cases}$$

The semi-trajectories γ^- and γ^+ of our system, issuing respectively from the points α_+ and α_- under the controls $v = -1$ and $v = 1$ and proceeding for $t \leq 0$, have the form

$$\gamma^- \sim \eta = \frac{1}{8}(7\xi^1 - 4\xi^2 - 6\xi^2 - 12\xi - 1),$$

$$\gamma^+ \sim \eta = \frac{1}{8}(-7\xi^1 - 4\xi^2 + 6\xi^3 - 12\xi + 1).$$

The reachable set is the region bounded by the curve γ , consisting of the semi-trajectories γ^+ and γ^- . We see from Example 3 that for nonlinear systems the reachable set can be non-convex.

The author expresses deep gratitude to their scientific supervisor L. N. Yeshukov for help and attention to the work.

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Figure 4: Figure 4