LINEAR TRANSFORMATIONS OF UNBOUNDED COMPLEX SEQUENCES

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Abstract

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LINEAR TRANSFORMATIONS OF UNBOUNDED COMPLEX SEQUENCES

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Let G be an unbounded domain of the complex plane. A sequence $\{s_n\}$ tends to ∞ in the domain G, which is denoted by

$$s_n \to \infty G,$$

if: 1) $s_n \in G$ for $n > N$, where N in general depends on $\{s_n\}$ and G; 2) $\lim_{n\to\infty} |s_n| = \infty$. The domain G itself in this case will be called the domain of convergence of the sequence $\{s_n\}$ to ∞.

Let (a_{mn}) $(m, n = 1, 2, ...)$ be a regular matrix with complex entries, and let $\{\sigma_m\}$ be the sequence obtained as the result of the linear transformation of the sequence $\{s_n\}$ by the matrix (a_{mn}):

$$\sigma_m = \sum_{n=1}^{\infty} a_{mn} s_n \quad (m = 1, 2, ...). \tag{1}$$

The following question is posed: what conditions must the matrix (a_{mn}) satisfy so that from the convergence $s_n \to \infty G$ there follows the convergence $\sigma_m \to \infty G'$, where G and G' are certain unbounded domains. In this formulation of the question, the concept of a fully regular transformation, i.e. a transformation for which from $s_n \to +\infty$ there follows $\sigma_m \to +\infty$, is in a certain sense carried over to complex sequences and matrices. Obviously, in our case the conditions on the matrix (a_{mn}) will depend on the form of the domains G and G' (or on the parameters determining these domains). Some cases in which the domains of convergence of the sequences $\{s_n\}$ and $\{\sigma_m\}$ to ∞ were angles and half-strips were considered in ([1]). In the present note more general results are given, pertaining to domains of the indicated types, as well as to arbitrary convex domains.

We first recall some definitions and notation. By $\Phi = \Phi(z, \alpha, \varphi)$ is denoted an angle with vertex at the point z, direction α^*, and magnitude φ $(0 \le \varphi \le 2\pi)$.

By $\Pi \equiv \Pi(z, \alpha, h)$ is denoted a half-strip of width (base) h, direction α, and with the midpoint of its base at the point z. A ray is regarded as an angle for $\varphi = 0$ or a half-strip for $h = 0$. By G_r is denoted the domain consisting of the domain G with the addition to it of all points whose distance from G is not greater than r. All domains are regarded as closed. If for the sequence $\{s_n\}$ the convergence $s_n \to \infty G_r$ takes place for every $r > 0$, then we shall say that $\{s_n\}$ converges asymptotically to ∞ in the domain G and denote this by $s_n \to \infty$ as G. The matrix (a_{mn}) will everywhere be assumed regular and row-finite.

We shall first give results showing that the linear transformation (1) cannot contract or shift the domain of convergence of each sequence. This follows from the following theorem:

* As the direction of the angle Φ we take the direction of the bisector of the angle; as the direction of the half-strip Π, the direction of the rays bounding the half-strip.

Theorem 1. Let G be an arbitrary unbounded domain. Whatever regular matrix (a_{mn}) is taken, it is always possible to construct a sequence $\{s_n\}$, $s_n \to \infty G$, such that for the sequence $\{\sigma_m\}$ convergence $\sigma_m \to \infty$ as G over this same domain G will take place.

From this theorem there follows

Corollary. Let G and G' be two arbitrary unbounded closed domains, and suppose that outside G' there lies an infinite part of G. There is no regular matrix (a_{mn}) which would transform every sequence $\{s_n\}$ converging to ∞ in the domain G into a sequence $\{\sigma_m\}$ converging to ∞ in the domain G'.

From the results stated it follows that, in studying conditions under which from the convergence $s_n \to \infty G$ there always follows the convergence $\sigma_m \to \infty G'$, it is of interest to consider the case when $G \subset G'$.

In the following theorems, conditions are established on the transformation matrix for certain kinds of domains. By $\widehat{a_0 a_1}$ is denoted the magnitude of the (smallest) angle through which a ray with direction a_0 must be turned in order to coincide with the ray having direction a_1; the angle $\widehat{a_0 a_1}$ is considered positive if this rotation is made counterclockwise, and negative in the opposite case. By $\arg z$ is denoted the principal value of the argument of the complex number z: $-\pi < \arg z \leq \pi$, and, by definition, $\arg 0 = 0$.

Theorem 2. Let $\Phi(z_0, \alpha_0, \varphi_0)$ and $\Phi(z_1, \alpha_1, \varphi_1)$ be fixed angles, $0 \leq \varphi_0 \leq \varphi_1 < 2\pi$, $\beta = \widehat{\alpha_0 \alpha_1}$. The condition

$$|\arg a_{mn} - \beta| \leq (\varphi_1 - \varphi_0)/2 \qquad (m > m_0,\ n > n_0) * \tag{2}$$

is necessary in order that from the convergence

$$s_n \to \infty\, \Phi(z_0, \alpha_0, \varphi_0) \tag{3}$$

there should always follow the convergence

$$\sigma_m \to \infty\, \Phi_r(z_1, \alpha_1, \varphi_1) \tag{4}$$

at least for one $r \geq 0$, in general depending on $\{s_n\}$; for $0 \leq \varphi_0 \leq \varphi_1 < \pi$ condition (2) is also sufficient in order that from convergence (3) there should follow convergence (4) for every $r > |z_1 - z_0|$.

Let us note that condition (2) for $0 \leq \varphi_0 \leq \varphi_1 < 2\pi$ is no longer sufficient for (4) to follow from (3) for some r (see Theorem 6 below).

Let us mention one corollary of this theorem.

Corollary. Let $\Phi(z_0, \alpha_0, \varphi_0)$ and $\Phi(z_0, \alpha_0, \varphi_1)$ $(0 \leq \varphi_0 \leq \varphi_1 < \pi)$ be fixed angles with common vertex z_0 and one and the same direction α_0. In order that from the convergence $s_n \to \infty\, \Phi(z_0, \alpha_0, \varphi_0)$ there should always follow the convergence $\sigma_m \to \infty$ as $\Phi(z_0, \alpha_0, \varphi_1)$, it is necessary and sufficient that the matrix (a_{mn}) satisfy the condition:

$$|\arg a_{mn}| \leq \frac{\varphi_1 - \varphi_0}{2} \qquad (m > m_0,\ n > n_0).$$

For $\varphi_0 = \varphi_1$ we obtain Theorem 6 from (1), and for $z_0 = \alpha_0 = \varphi_0 = \varphi_1 = 0$, the conditions of "weakened" complete regularity.

In the following theorem the position of the angle of convergence of the sequence $\{\sigma_m\}$ to ∞ is not fixed in advance, but only it is required that its magnitude not exceed the given value φ_1.

Theorem 3. Let $\Phi(z_0, \alpha_0, \varphi_0)$ be a fixed angle and $0 \leq \varphi_0 \leq \varphi_1 < 2\pi$. If the matrix (a_{mn}) has the property that from the converg–

* That is, there exist natural numbers m_0 and n_0 such that for $m > m_0,\ n > n_0$ the indicated inequality holds; this is meant everywhere below.

of the convergence $s_n \to \infty\, \Phi(z_0, \alpha_0, \varphi_0)$ there always follows the convergence $\sigma_m \to \infty\, \Phi(z, \alpha, \varphi)$ for some z, α and φ, $\varphi \leq \varphi_1$, in general depending on $\{s_n\}$, then there necessarily exists such a direction α_1 that, for any sequence $\{s_n\}$ satisfying the condition $s_n \to \infty\, \Phi(z_0, \alpha_0, \varphi_0)$, there will hold the convergence $\sigma_m \to \infty\, \Phi_r(z_0, \alpha_1, \varphi_1)$ for some $r > 0$, in general depending on $\{s_n\}$.

From Theorems 2 and 3 it follows that

Corollary. Let $\Phi(z_0, \alpha_0, \varphi_0)$ be a fixed angle and let $0 \leq \varphi_0 \leq \varphi_1 < 2\pi$. In order that from the convergence $s_n \to \infty\, \Phi(z_0, \alpha_0, \varphi_0)$ there always follow the convergence $\sigma_m \to \infty\, \Phi(z, \alpha, \varphi)$ for some z, α and φ, $\varphi \leq \varphi_1$, in general depending on $\{s_n\}$, it is necessary that there exist a real number β such that

$$|\arg a_{mn} - \beta| \leq (\varphi_1 - \varphi_0)/2 \qquad (m > m_0,\ n > n_0). \tag{5}$$

For $0 \leq \varphi_0 \leq \varphi_1 < \pi$ this condition is also sufficient for the convergence $\sigma_m \to \infty$ as $\Phi(z_0, \alpha_1, \varphi_1)$ to take place, where $\alpha_1 = \alpha_0 + \beta$.

If, for the magnitude φ of the angle of convergence of the sequence $\{\sigma_m\}$, the restriction $\varphi \leq \varphi_1 < \pi$ is replaced simply by the inequality $\varphi < \pi$, then the conditions on the transformation matrix change, as the following theorem shows.

Theorem 4. Let $\Phi(z_0, \alpha_0, \varphi_0)$ $(\varphi_0 < \pi)$ be a fixed angle. In order that from the convergence $s_n \to \infty\, \Phi(z_0, \alpha_0, \varphi_0)$ there always follow the convergence $\sigma_m \to \infty\, \Phi(z, \alpha, \varphi)$ for some z, α, φ $(\varphi < \pi)$, in general depending on $\{s_n\}$, it is necessary and sufficient that there exist a real number β such that

$$\varlimsup_{\substack{m\to\infty \\ n\to\infty}} |\arg a_{mn} - \beta| = C < (\pi - \varphi_0)/2.$$

In the following theorem, conditions on the matrix are established in the case when, for the magnitude of the angle of convergence of the sequence $\{\sigma_m\}$ to ∞, an ε-enlargement relative to a fixed value φ_1 is allowed.

Theorem 5. Let $\Phi(z_0, \alpha_0, \varphi_0)$ and $\Phi(z_1, \alpha_1, \varphi_1)$ $(0 \leq \varphi_0 \leq \varphi_1 < \pi)$ be fixed angles, $\beta = \widehat{\alpha_0 \alpha_1}$. In order that from the convergence $s_n \to \infty\, \Phi(z_0, \alpha_0, \varphi_0)$ there always follow the convergence $\sigma_m \to \infty\, \Phi(z_1, \alpha_1, \varphi_1 + \varepsilon)$ for every $\varepsilon > 0$ $(\varphi_1 + \varepsilon < \pi)$, it is necessary and sufficient that

$$\varlimsup_{\substack{n\to\infty \\ m\to\infty}} |\arg a_{mn} - \beta| \leq (\varphi_1 - \varphi_0)/2.$$

Hence, as a special case, for $z_0 = z_1,\ \alpha_0 = \alpha_1,\ \varphi_0 = \varphi_1$, Theorem 7 from [1] follows.

For angles whose magnitudes φ_0 and φ_1 are subject to the condition $\pi \leq \varphi_0 \leq \varphi_1 < 2\pi$, the following result holds. By $(a_{mn})_p$ is denoted the matrix obtained from the matrix (a_{mn}) by replacing with zeros all elements for which either $m < p$ or $n < p$.

Theorem 6. Let $\Phi(z_0, \alpha_0, \varphi_0)$ be a fixed angle and let $\pi \leq \varphi_0 \leq \varphi_1 < 2\pi$. If the matrix (a_{mn}) has the property that from the convergence $s_n \to \infty\, \Phi(z_0, \alpha_0, \varphi_0)$ there always follows the convergence $\sigma_m \to \infty\, \Phi(z, \alpha, \varphi)$ for some $z, \alpha, \varphi,\ \varphi \leq \varphi_1$, in general depending on $\{s_n\}$, then it necessarily satisfies condition (5) for some β, and, moreover, there exists such a p that the matrix $(a_{mn})_p$ contains in each row no more than one element different from zero.

It follows from this theorem that if a regular matrix (a_{mn}) has the property formulated in the theorem, then in fact in this case from the convergence $s_n \to \infty\, \Phi(z_0, \alpha_0, \varphi_0)$ there will always follow the convergence $\sigma_m \to \infty\, \Phi(z_0, \alpha_0, \varphi_0 + \varepsilon)$ for every $\varepsilon > 0$, and, moreover, for $\varphi_0 = \varphi_1$ the matrix (a_{mn}) must satisfy the condition $\arg a_{mn} = 0$ $(m > m_0,\ n > n_0)$, and in this case the convergence $\sigma_m \to \infty$ as $\Phi(z_0, \alpha_0, \varphi_0)$ will take place.

The following result concerns the transformation of sequences converging to ∞ in a half-strip into sequences converging to ∞ in an angle (with an r-extension).

Theorem 7. Let $\Pi(z_0, \alpha_0, h_0)$ and $\Phi(z_1, \alpha_1, \varphi_1)$ $(\varphi_1 < \pi)$ be a fixed half-strip and angle. The condition

$$|\arg a_{mn} - \beta| \le \varphi_1/2 \qquad (m > m_0,\ n > n_0),$$

where $\beta = \widehat{\alpha_0\alpha_1}$, is necessary in order that from the convergence

$$s_n \to \infty\ \Pi(z_0, \alpha_0, h_0) \tag{6}$$

there should always follow the convergence

$$\sigma_m \to \infty\ \Phi_r(z_1, \alpha_1, \varphi_1) \tag{7}$$

for at least one $r \ge 0$, and is sufficient in order that (7) follow from (6) for every $r > r_1$, where

$$r_1 = |z_1 - z;| + \frac{h_0}{2} \sup_m \sum_{n=1}^{\infty} |a_{mn}|.$$

For $\varphi_1 = 0$, this theorem yields conditions on the matrix for transforming a sequence $\{s_n\}$, converging to ∞ in a half-strip, into a sequence $\{\sigma_m\}$ also converging to ∞ in a half-strip. By virtue of the regularity of the matrix (a_{mn}), in this case it is necessary that $\beta = 0$.

The following theorem concerns transformations of sequences converging to ∞ in a convex domain.

Theorem 8. Let G be an arbitrary convex domain contained in some angle smaller than π. If for every such domain the convergence

$$s_n \to \infty\ G \tag{8}$$

always implies the convergence

$$\sigma_m \to \infty\ G_r \tag{9}$$

for at least one $r \ge 0$, in general depending on $\{s_n\}$, then the matrix (a_{mn}) must satisfy the conditions:

1) $\arg a_{mn} = 0$ $(m > m_0,\ n > n_0)$,
2) there exist numbers M and N such that

$$\sum_{n=N}^{\infty} a_{mn} \leq 1 \quad \text{for all } m > M.$$

These conditions are sufficient in order that (9) follow from (8) for every $r > 0$ (i.e., in order that the convergence $\sigma_m \to \infty$ as G hold).

Let us note that the necessity of conditions 1) and 2) in this theorem is understood in the sense that, if either of them is violated, then there exists a convex domain G, contained in some angle smaller than π, and a sequence $\{s_n\}$ satisfying condition (8), for which convergence (9) does not occur for any r.

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CITED LITERATURE

1. I. I. Volkov, DAN, 165, No. 4, 742 (1965).

Note: Figure translations are in progress. See original paper for figures.