THE SHIFT OPERATOR ALONG TRAJECTORIES OF EVOLUTION EQUATIONS AND PERIODIC SOLUTIONS

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Abstract

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THE SHIFT OPERATOR ALONG TRAJECTORIES OF EVOLUTION EQUATIONS AND PERIODIC SOLUTIONS

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1. The article considers linear and nonlinear systems of differential equations of the form

$$x'(t) = f(t, x_t), \tag{1}$$

where $x_t = x(t+s)$, $-\omega \leq s \leq 0$. Here $f(t, x_0) = (f_1, f_2, \dots, f_n)$, and each component $f_i(t, x_0(s))$ is a functional on elements $x_0(s) = (x_{01}(s), \dots, x_{0n}(s))$ of the space $C^n[-\omega, 0]$ of continuous vector-functions. System (1) includes, for example, systems of integro-differential equations with retarded argument of the form

$$x'(t) = F(t, x(t), x(t-h)) + \int_{-H}^{0} G(t, s, x(t), x(t+s))\, ds, \quad h, H \leq \omega, \tag{1'}$$

and in essence is a convenient notation for such systems. We assume these functionals to be defined on some sphere $T_a = (\|x\| < a)$, continuous in x_0 uniformly with respect to (t, x_0), and bounded. We also assume measurability of the superposition $f(t, x_t)$, if $x(t)$ is measurable (superpositional measurability).

We shall consider the problem of periodic solutions of system (1), assuming the right-hand side to be ω-periodic. If the Cauchy problem $x_{t=0} = x_0(s)$ is uniquely and nonlocally solvable for any initial function from some closed domain $\overline{\Omega} \subset T_a$, then the shift operator is defined by

$$\Pi_t x_0 = x_t, \quad t \geq 0, \quad x_0 \in \overline{\Omega}, \tag{2}$$

where $x(t)$ is the solution of the Cauchy problem generated by the initial function $x_0(s)$. It is well known that the fixed points of the operator Π_ω, and only they, generate ω-periodic solutions of system (1) that start in the domain $\overline{\Omega}$ (see, for example, ([1,2])). In ([2,3]), topological methods were applied to the study of the operator Π_ω.

In the present article we shall apply methods of cone theory ([4,5]).

2. Let us recall some notions. Denote by K the cone of nonnegative functions in the space $C[-\omega, 0]$. By a wedge W_m in the space $C^n[-\omega, 0]$ we shall mean the set of those vector-functions $x_0(s)$ which satisfy the condition $x_{0k}(s) \in K$ for all $k = 1, \dots, m$ $(m \leq n)$. In particular, W_n is the cone K_n of nonnegative vector-functions. We shall call the elements of the wedge **semipositive**, and the elements of the cone **positive**.

Let the domain $\overline{\Omega}$ belong to W_m. We shall call the operator Π_ω **semipositive** on $\overline{\Omega}$ if $\Pi_\omega \overline{\Omega} \subset W_m$. In particular, when $W_n = K_n$, the semipositivity of the operator Π_ω means its positivity on $\overline{\Omega}$.

Let, for every $t \in [0, \omega]$ and every $k = 1, \dots, m$,

$$f_k(t, x_0(s)) \geq 0, \quad \text{if } x_{0k}(0) = 0, \ x_0(s) \in W_m. \tag{3}$$

Theorem 1. *If conditions No. 1 and condition (3) are satisfied, then the operator Π_ω is semipositive on $\overline{\Omega}$ and completely continuous.*

Below, the conditions of Theorem 1 are always assumed to be fulfilled.

A completely continuous semipositive operator Π_ω generates on the boundary $\dot{\Omega}$ of the open domain Ω (in the relative topology W_m) the vector field

$$\Phi x_0 = \Pi_\omega x_0 - x_0,$$

directed into the wedge W_m, and if this field has no singular points, then its rotation $\gamma\{\Phi, \dot{\Omega}, W_m\}$ relative to the wedge W_m is defined [5]; the rotation of the vector field is a topological invariant equal to the algebraic number of fixed points of the operator Π_ω lying in Ω.

The problem of computing the rotation γ is difficult. The general principle of Leray for a homotopic transformation of a vector field into a simpler one, in our situation, can be formulated as follows.

Suppose that in equation (1) one can introduce a numerical parameter λ, $0 \leq \lambda \leq 1$, so that the following conditions are satisfied:

1) $f(t, x_0, \lambda)$ satisfies condition (3) for all λ;
2) for all λ the system
$$x'(t) = f(t, x_t, \lambda) \tag{4}$$
satisfies conditions No. 1 and the operator $\Pi_\omega(x_0, \lambda)$ is completely continuous jointly in (x_0, λ);
3) $f(t, x_0, 1) = f(t, x_0)$;

4) $f(t, x_0, 0) = f_0(t, x_0(0))$, where $f_0(t, c)$ is a function of finitely many variables (t, c) ($c \in E^n$, finite-dimensional Euclidean space).

Under these conditions we shall say that system (1) is homotopic to the system of ordinary differential equations

$$x'(t) = f_0(t, x(t)). \tag{5}$$

Denote by $\Pi^0_\omega x_0$ the shift operator for system (5) in the finite-dimensional space E^n, and denote the corresponding vector field by $\Phi^0 x_0$. Define in E^n the wedge W^n_m as the set of vectors $x_0 = (x_{01}, \dots, x_{0n})$ whose first m coordinates are nonnegative, and let Ω^n be some open subset of the wedge W^n_m.

It turns out that the rotations of the vector fields Φ and Φ^0 on the boundaries $\dot\Omega$ and $\dot\Omega^n$ are related to one another for an appropriate choice of the domains Ω, Ω^n. Connections of this kind were first noted in the works [2,6] (in [6], in a more general form for domains in $C^n[-\omega, 0]$). We shall generalize the result of [2] for the operator Π_ω acting in the wedge W_m.

Theorem 2. *Let the set $\mathfrak{M}$ of semipositive ω-periodic solutions of system (4) be bounded for all λ, and let Ω^n and Ω be relative spheres in the wedges W^n_m, W_m, respectively, with center at the point θ and containing the set $\mathfrak{M}$. Then*

$$\gamma\{\Phi, \dot\Omega, W_m\} = \gamma\{\Phi^0, \dot\Omega^n, W^n_m\}.$$

In the case when equation (1) is linearized at zero or at infinity, the rotation of the field Φ on relative spheres $\dot\Omega_0$ or $\dot\Omega_\infty$ of sufficiently small, respectively large, radius is computed from the corresponding linear systems

$$x' = L_0(t, x_t), \qquad x' = L_\infty(t, x_t),$$

provided that they have no semipositive ω-periodic solutions other than zero. In this case the formula is valid

$$\gamma\{\Phi, \dot\Omega_0, W_m\} = \gamma_0(\theta), \qquad \gamma\{\Phi, \dot\Omega_\infty, W_m\} = \gamma_\infty(\theta),$$

where $\gamma_0(\theta)$, $\gamma_\infty(\theta)$ are the indices of the point θ of the shift operators of the linear systems.

3. The theorem 3 stated below is based on the computation of the indices $\gamma_0(\theta)$, $\gamma_\infty(\theta)$. We represent the elements of the wedge $x_0(s)$ in the form $x_0 = (y_0, z_0)$, where y_0 are the nonnegative components of the vector x_0, forming the cone K_m,

and z_0 the remaining ones. Then for the functionals L_0, L_∞, by condition (3), we have the decomposition

$$L_k(t, x_0) = \big(L_{k1}(t, y_0),\, L_{k2}(t, y_0, z_0)\big) \quad (k = 0, \infty).$$

Consider the linear systems

$$y'(t) = L_{k1}(t, y_t), \tag{6}$$

$$z'(t) = L_{k2}(t, 0, z_t) \quad (k = 0, \infty), \tag{7}$$

and denote the fixed-point indices of the corresponding translation operators by γ_{ki}, $i = 1, 2$ (γ_{k1} relative to the cone K_m, and γ_{k2} relative to the space $C^{n-m}[-\omega, 0]$).

Theorem 3. *Suppose that: 1) the functional $f(t, x_0)$ has the asymptotic Fréchet derivative; 2) for $k = \infty$ system (7) has no nonzero ω-periodic solutions, and system (6) has no nonzero positive ω-periodic solutions. Finally, let $\gamma_{\infty 1} \cdot \gamma_{\infty 2} \neq 0$. Then system (1) has a semipositive ω-periodic solution.*

Suppose, moreover, that the functional $f(t, x_0)$ has a Fréchet derivative at zero, $f(t, \theta) \equiv 0$, and that condition 2) is satisfied for systems (7), (6) ($k = 0$). Let $\gamma_{01} \cdot \gamma_{02} \neq \gamma_{\infty 1} \cdot \gamma_{\infty 2}$. Then system (1) has, in addition to the zero solution, a nonzero ω-periodic semipositive solution.

Below, for simplicity, we shall omit the index k in the notation.

We note that the index $\gamma_2(\theta)$ can be computed by the formula $\gamma_2(\theta) = (-1)^{\beta}$, where β is the sum of the orders of the eigenvalues of the translation operator for system (7) that are greater than one ([7]). Similarly, the index $\gamma_1(\theta)$ can be related to the positive spectrum of the translation operator A_ω of system (6): if A_ω is a strongly positive operator, then it has a unique simple eigenvalue λ_1^+ ([4]), and $\gamma_1(\theta) = 1$ if $\lambda_1^+ < 1$, and $\gamma_1(\theta) = 0$ if $\lambda_1^+ > 1$ ([5]).

Let us formulate a sufficient condition for the strong positivity of the operator A_ω. Let system (6) have the form

$$y'(t) = S(t)y(t) + R(t, y_t), \tag{6'}$$

where $S(t)$ is an ω-periodic matrix, and $R(t, y_0)$ is a nonnegative functional on K_m. Under these conditions the following is true.

Theorem 4. *If the translation operator D_ω of the system of ordinary differential equations $y' = S(t)y$ is strongly positive on the cone of nonnegative vectors $y \geq \theta$, and the vector-function $R(t) = R(t, y_t)$ is not identically zero on the interval $[0, \omega]$ for any solution $y(t)$ of system (6) with a nonzero initial condition $y_0(s) \geq 0$, $y_0(0) = 0$, then the operator A_ω is strongly positive.*

We note that an analogous assertion also holds for nonlinear systems.

Sufficient conditions for the strong positivity of the operator D_ω are found in work ([8]).

4. Let us compute the indices γ_i in terms of the right-hand side of systems (6), (7). Let systems (6), (7) be stationary, i.e.,

$$y'(t) = L_1(y_t), \qquad z'(t) = L_2(0, z_t). \tag{8}$$

Then, to compute the indices $\gamma_1(\theta), \gamma_2(\theta)$ of systems (8), one may use the spectrum of the infinitesimal generators A_t, B_t, $t \geq 0$ ([9–11]), generated by equations (8).

Consider the semigroup A_t. Its infinitesimal generator $Ay_0(s) = dy_0/ds$ has as its domain continuously differentiable functions satisfying the boundary condition $y_0' = L_1(y_0)$. The complex spectrum of the operator A is determined by the characteristic equation (for example, see ([11]))

$$\left|\widetilde{L}_1(e^{\mu s}) - \mu I\right| = 0. \tag{9}$$

Here $\widetilde{L}_1(e^{\mu s})$ is a matrix of order m, whose j-th column has the form $L_1(y_0^j(s))$, where $y_0^j(s)$ is a vector-function with zero components except for the j-th, equal to $e^{\mu s}$.

Theorem 5. *Let A_ω be a strongly positive operator. Then there exists a unique real root μ_0 of equation (9) such that the system of algebraic equations $(\widetilde{L}_1(e^{\mu_0 s}) - \mu_0 I)c = 0$ has a unique normalized strictly positive solution c_+. Moreover, $\gamma_1(\theta) = 1$ if $\mu_0 < 0$, and $\gamma_1(\theta) = 0$ if $\mu_0 > 0$. (By a strictly positive solution we mean a solution all of whose components are positive.)*

Let us note that $\mu_0 > \operatorname{Re}\mu$, where μ is any root of equation (9).

As an example, consider the second-order system

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = A\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + B\begin{pmatrix} x(t-h) \\ y(t-h) \end{pmatrix} + \int_{-h}^{0} \beta(s)\begin{pmatrix} x(t+s) \\ y(t+s) \end{pmatrix} ds.$$

Here $A = (a_{ij})$, $B = (b_{ij})$ are constant matrices, and the matrix $\beta(s) = (\beta_{ij}(s))$ is summable and nonnegative for all s. Let $a_{12}, a_{21} > 0$, $b_{ij} \geq 0$, and suppose that at least one of the matrices B, $C = \int_{-h}^{0} \beta(s)\, ds$ has at least one strictly positive row.

Denote

$$\Delta = (a_{11} + b_{11} + c_{11})(a_{22} + b_{22} + c_{22}) - (a_{12} + b_{12} + c_{12})(a_{21} + b_{21} + c_{21}),$$

and denote by $\gamma(\theta)$ the index of the point θ of the shift operator of our system, computed with respect to the cone of nonnegative functions (x, y).

Theorem 6. *Let the conditions listed above be satisfied, and let*

$$a_{11} < 0, \qquad a_{22} < 0, \qquad |a_{11}| > b_{11} + c_{11}, \qquad |a_{22}| > b_{22} + c_{22}. \tag{10}$$

Then: 1) $\gamma(\theta) = 1$, if $\Delta > 0$; 2) $\gamma(\theta) = 0$, if at least one of the conditions (10) is not fulfilled, or $\Delta < 0$; 3) $\gamma(\theta)$ is undefined if $\Delta = 0$.

To compute the index $\gamma_2(\theta)$, note that the root subspaces of the infinitesimal operator B of the semigroup B_t are normally split off in the sense of (12), and the operator $B-\mu I$ is normally solvable and has finite defect numbers. Let the eigenvalues μ of the operator B be distinct from $2\pi k\omega^{-1}i$, $k=0,\pm1,\ldots$. Using the relation between the spectra and root subspaces of the operators B and $\bar{B}_\omega$, we obtain $\gamma_2(\theta)=(-1)^\beta$, where β is the sum of the orders of the positive eigenvalues μ of the infinitesimal operator B.

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Note: Figure translations are in progress. See original paper for figures.

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