NONLINEAR ABSTRACT HYPERBOLIC EQUATIONS\

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Abstract

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NONLINEAR ABSTRACT HYPERBOLIC EQUATIONS*

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The paper investigates the solvability of the Cauchy problem for nonlinear abstract hyperbolic equations of second order. Local and nonlocal existence theorems are proved. The abstract results obtained are applied to the study of a boundary-value problem for one class of nonlinear hyperbolic equations in partial derivatives.

1. Consider the problem

$$d^2u(t)/dt^2 + A(t)u(t) = f(t, u(t), du(t)/dt), \quad u(0) = u_0, \quad u'(0) = u_1 \tag{1}$$

in a Hilbert space H. By a solution of problem (1) we mean a function $u(t)$, twice continuously differentiable on $[0, T]$, satisfying equation (1) for every $t \in [0, T]$ and having the property that the functions $A(t)u(t)$, $A^{1/2}(t)du(t)/dt$ are continuous on $[0, T]$. The investigation of problem (1) is carried out by methods of the theory of semigroups ([1]).

Let $A(0) = A_0$ be a self-adjoint positive definite operator in H with domain of definition $D(A)$. Define the Hilbert space $H(A^\alpha)$, consisting of elements $D(A^\alpha)$ with scalar product $(u, v)_{A^\alpha} = (A_0^\alpha u, A_0^\alpha v)$.

Theorem 1. *Suppose that the following conditions are fulfilled:*

1°. *For each $t \in [0, T]$ the operator $A(t)$ is self-adjoint positive definite in H, and for all $t \in [0, T]$ and all $u \in D(A(t))$ one has*

$$(A(t)u, u) \geq \gamma(u, u) \quad (\gamma = \text{const} > 0).$$

2°. *The operator $A^{1/2}(t)$ has a domain of definition $D(A^{1/2})$ independent of t; the operator-function $A^{1/2}(t)A^{-1/2}(0)$ is twice strongly continuously differentiable on $[0, T]$**.*

3°. $u_0 \in D(A(0))$, $u_1 \in D(A^{1/2})$.

4°. *The operator $f(t,u,v)$ acts from $[0,T]\times H(A^{1/2})\times H(A^{1/2})$ into H and is bounded; for any function $u(t)$ continuously differentiable in $H(A^{1/2})$ and any function $v(t)$, continuous in $H(A^{1/2})$ and continuously differentiable in H, the vector-function $f(t,u(t),v(t))$ is continuously differentiable in H; for any two pairs of functions $u_1(t),u_2(t),v_1(t),v_2(t)$, respectively satisfying these smoothness conditions, the inequality holds*

$$\left\|\frac{d}{dt}\left[f(t,u_1(t),v_1(t))-f(t,u_2(t),v_2(t))\right]\right\|\le$$

$$\le c(R)\left[\left\|A_0^{1/2}(u_1(t)-u_2(t))\right\|+\left\|A_0^{1/2}(u_1'(t)-u_2'(t))\right\|+\right.$$

$$\left.+\left\|A_0^{1/2}(v_1(t)-v_2(t))\right\|+\|v_1'(t)-v_2'(t)\|\right],$$

as soon as

$$\left\|A_0^{1/2}u_i(t)\right\|+\left\|A_0^{1/2}u_i'(t)\right\|\le R,\quad \left\|A_0^{1/2}v_i(t)\right\|+\|v_i'(t)\|<R\quad (i=1,2).$$

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* The main part of the paper was reported at the All-Union Symposium "Some Problems of the Theory of Differential and Integral Equations" in April 1964, Dushanbe.

** Twice strong continuous differentiability of the operator-function $A^{1/2}(t)A^{-1/2}(0)$ may be replaced by the condition of continuity of the operator-function $A^{1/2}(0)(A^{1/2}(t))'A^{-1}(0)$.

Then problem (1) has a unique solution on some interval $[0,t_0]\subset[0,T]$, which can be found by the method of successive approximations.

Theorem 2. Suppose that the hypotheses of Theorem 1 are satisfied. Suppose, further:

5°. For any functions $u(t)$ that are continuous in $H(A^{1/2})$ and twice continuously differentiable in H, the following holds:

$$\operatorname{Re}\int_0^t (f(\tau,u(\tau),u'(\tau)),u'(\tau))\,d\tau\le C\left[1+\int_0^t\left(\|A_0^{1/2}u(\tau)\|^2+\|u'(\tau)\|^2\right)d\tau\right];$$

for all functions $u(t)$ satisfying these smoothness conditions, the inequality

$$\left\|\frac{d}{dt}f(t,u(t),u'(t))\right\|\le C(R)\,[1+\|A_0^{1/2}u'(t)\|+\|u''(t)\|],$$

holds as soon as

$$\|A_0^{1/2}u(t)\|+\|u'(t)\|\le R.$$

Then problem (1) has a unique solution on $[0, T]$.

In the case when not only $D(A^{1/2}(t))$, but also $D(A(t))$ does not depend on t, stronger theorems are proved.

Theorem 3. Suppose that hypotheses 1° and 3° of Theorem 1 are satisfied. Suppose, further:

6°. The operator $A(t)$ has a domain of definition $D(A)$ independent of t; the operator-valued function $A(t)A^{-1}(0)$ is strongly continuously differentiable.

7°. The operator $f(t, u, v)$ maps $[0, T] \times H(A) \times H(A^{1/2})$ into H and is bounded; for any function $u(t)$ continuous in $H(A)$ and continuously differentiable in $H(A^{1/2})$, and any function $v(t)$ continuous in $H(A^{1/2})$ and continuously differentiable in H, the vector-valued function $f(t, u(t), v(t))$ is continuously differentiable in H; for any two pairs of functions $u_1(t), u_2(t), v_1(t), v_2(t)$, respectively satisfying these smoothness conditions, the inequality

$$\left\| \frac{d}{dt} \left[f(t, u_1(t), v_1(t)) - f(t, u_2(t), v_2(t)) \right] \right\| \leq$$

$$\leq C(R) \left[\|A_0(u_1(t) - u_2(t))\| + \|A_0^{1/2}(u_1'(t) - u_2'(t))\| + \right.$$

$$\left. + \|A_0^{1/2}(v_1(t) - v_2(t))\| + \|v_1'(t) - v_2'(t)\| \right],$$

holds as soon as

$$\|A_0 u_i(t)\| + \|A_0^{1/2} u_i'(t)\| \leq R, \quad \|A_0^{1/2} v_i(t)\| + \|v_i'(t)\| \leq R$$

$$(i = 1, 2).$$

Then problem (1) has a unique solution on some interval $[0, t_0] \subset [0, T]$, which can be found by the method of successive approximations.

Theorem 4. Suppose that hypotheses 1° and 3° of Theorem 1 and hypotheses 6° and 7° of Theorem 3 are satisfied. Suppose, further:

8°. For any functions $u(t)$ continuous in $H(A)$, continuously differentiable in $H(A^{1/2})$, and twice continuously differentiable in H, the following holds:

$$\operatorname{Re} \int_0^t (f(\tau, u(\tau), u'(\tau)), u'(\tau)) \, d\tau \leq C \left[1 + \int_0^t \left(\|A_0^{1/2} u(\tau)\|^2 + \|u'(\tau)\|^2 \right) d\tau \right];$$

for all functions $u(t)$ satisfying these smoothness conditions, the inequality

$$\left\| \frac{d}{dt} f(t, u(t), u'(t)) \right\| \leq C(R) [1 + \|A_0 u(t)\| + \|A_0^{1/2} u'(t)\| + \|u''(t)\|],$$

holds as soon as

$$\|A_0^{1/2} u(t)\| + \|u'(t)\| \leq R.$$

Then problem (1) has a unique solution on $[0,T]$.

In the case when the operator A does not depend on t, Theorems 3 and 4 strengthen the corresponding results of works $(2,3)$. Let us also note that the theorems

1–4 strengthen and supplement some results obtained earlier by the author (^{4,5}). Nonlocal existence theorems for problem (1) under other assumptions and in another class of solutions were obtained in (^{6}).

2. Consider the problem:

$$d^2u(t)/dt^2 + A(t,u(t))u(t) = f(t,u(t)), \qquad u(0) = u_0, \qquad u'(0) = u_1 \qquad (2)$$

Apparently, hyperbolic equations of the form (1) have not been studied. With the aid of the special substitution

$$v_1(t) = \tfrac{1}{2}[iA^{1/2}(t,u(t))u(t) + u'(t)],$$

$$v_2(t) = \tfrac{1}{2}[-iA^{1/2}(t,u(t))u(t) + u'(t)],$$

problem (2) is reduced to the equivalent problem

$$dv(t)/dt = \mathfrak{A}_1(t,[Bv](t))v(t) + \mathfrak{A}_2(t,[Bv](t))v(t) +$$
$$+F(t,[Bv](t)), \qquad v(0) = v_0, \qquad (3)$$

where

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad v_0 = \begin{pmatrix} \frac{1}{2}[iA^{1/2}(0,u_0)u_0 + u_1] \\ \frac{1}{2}[-iA^{1/2}(0,u_0)u_0 + u_1] \end{pmatrix},$$

$$\mathfrak{A}_1(t,v) = \begin{pmatrix} iA^{1/2}(t,v_1) & 0 \\ 0 & -iA^{1/2}(t,v_1) \end{pmatrix},$$

$$\mathfrak{A}_2(t,v(t)) = \begin{pmatrix} \frac{1}{2}\dfrac{dA^{1/2}(t,v_1(t))}{dt}A^{-1/2}(t,v_1(t)) & -\frac{1}{2}\dfrac{dA^{1/2}(t,v_1(t))}{dt}A^{-1/2}(t,v_1(t)) \\ -\frac{1}{2}\dfrac{dA^{1/2}(t,v_1(t))}{dt}A^{-1/2}(t,v_1(t)) & \frac{1}{2}\dfrac{dA^{1/2}(t,v_1(t))}{dt}A^{-1/2}(t,v_1(t)) \end{pmatrix},$$

$$F(t,v) = \begin{pmatrix} \frac{1}{2}f(t,v_1) \\ \frac{1}{2}f(t,v_1) \end{pmatrix}, \qquad [Bv](t) = u_0 + \int_0^t (v_1(\tau) + v_2(\tau))\,d\tau,$$

in the Hilbert space $H^2 = H \times H$. In what follows, problem (3) is investigated by the method of "freezing" the coefficients.

Theorem 5. *Suppose that the following conditions are satisfied:*

1°. *For every $t \in [0,T]$, $u \in H$, the operator $A(t,u)$ is self-adjoint and positive definite in H, with domain of definition*

$$D(A(t,u)) = D(A);$$

for every $t \in [0,T]$ *the inequality holds*

$$(A(t,u)v,v) \geq \gamma(v,v), \qquad \|A(t,u)v\| \geq \gamma\|v\|_A.$$

2°. *The operator* $A(t,u)A_0^{-1}$ *acts from* $[0,T] \times H$ *into* $B(H)^*$ *and is bounded; for every continuously differentiable vector-function* $u(t)$ *the operator-function* $A(t,u(t))A_0^{-1}$ *is strongly continuously differentiable and*

$$\left\|\frac{d}{dt}A(t,u(t))A_0^{-1}\right\| \leq C(R)$$

as soon as $\|u(t)\| + \|du(t)/dt\| \leq R$*; for any pair of such functions* $u_1(t), u_2(t)$ *the inequality holds*

$$\left\|\frac{d}{dt}[A^{1/2}(t,u_1(t)) - A^{1/2}(t,u_2(t))]A_0^{-1/2}\right\| \leq$$

$$\leq C(R)\left[\|u_1(t) - u_2(t)\| + \left\|\frac{du_1(t)}{dt} + \frac{du_2(t)}{dt}\right\|\right],$$

3°. *The operator* $f(t,u)$ *acts from* $[0,T] \times H$ *into* H *and is bounded; for every continuously differentiable function* $u(t)$ *the vector-function* $f(t,u(t))$ *is continuously differentiable and*

$$\left\|\frac{d}{dt}f(t,u(t))\right\| \leq C(R),$$

as soon as $\|u(t)\| + \|du(t)/dt\| \leq R$*; for any pair of such functions* $u_1(t), u_2(t)$ *the inequality holds*

$$\|f(t,u_1(t)) - f(t,u_2(t))\| \leq C(R)[\|u_1(t) - u_2(t)\| +$$

$$+\|du_1(t)/dt - du_2(t)/dt\|].$$

4°. $u_0 \in D(A), \quad u_1 \in D(A^{1/2})$.

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* $B(H)$ is the space of bounded operators on H.

Then problem (2) has a unique solution on some interval $[0,t_0] \subset [0,T]$, which can be found by the method of successive approximations.

3. As an application, let us consider, in a bounded cylinder $Q = [0,T] \times \Omega$, where Ω is a domain of variation of $x = (x_1, \dots, x_n)$ in R^n $(n \leqslant 3)$, the first boundary-value problem for the quasilinear hyperbolic equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(t,x)\frac{\partial u(t,x)}{\partial x_j}\right) + b(t,x)\frac{\partial u(t,x)}{\partial t} =$$

$$= f(t, x, u(t, x), |u(t, x)|^2) \tag{4}$$

with the conditions

$$u(t, x)\big|_{t=0} = u_0(x), \qquad u_t'(t, x)\big|_{t=0} = u_1(x), \qquad u(t, x)\big|_{\Gamma} = 0. \tag{5}$$

Applying results ([7],[8]) and Theorem 3 of the present note, we prove the following theorem.

Theorem 6. Suppose that the following conditions are satisfied:

a) the coefficients a_{ij} are real, and

$$a_{ij} = a_{ji}, \qquad \sum_{i,j=1}^{n} a_{ij}(t, x)\xi_i\xi_j \geqslant \alpha \sum \xi_i^2,$$

where $\alpha = \text{const} > 0$, for all real $\xi_1, \ldots, \xi_n$; the functions a_{ij}, $\partial a_{ij}/\partial x_i$, and their first derivatives with respect to t are continuous in Q; the boundary Γ of the domain Ω is twice continuously differentiable;

b) the functions $b(t, x)$, $f(t, x, u, r)$ are continuous together with their derivatives with respect to t, u, r in the domain $\{0 \le t \le T,\ x \in \Omega,\ |u| \le R,\ 0 \le r \le R\}$, and these derivatives satisfy, with respect to u, r, a Lipschitz condition in each bounded set $|u| \le R,\ 0 \le r \le R$, with a constant depending only on R,

c) $u_0(x) \in W_{2,0}^2(\Omega), \qquad u_1(x) \in W_2^1(\Omega)$.

Then problem (4)–(5) has a unique solution in some cylinder.

The author's next article ([9]) is devoted to the application of the results obtained in the present article to quasilinear partial differential equations of hyperbolic type. Nonlocal existence theorems are proved.

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Note: Figure translations are in progress. See original paper for figures.

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