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PROGRESSIONS OF
THE (k) -TH ORDER
FOR CONSTRUCTING
BASES OF THE
ALGEBRA OF
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FUNCTIONS**

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Abstract

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MATHEMATICS

I. A. LAVROV

THE USE OF ARITHMETIC PROGRESSIONS OF THE k -TH ORDER FOR CONSTRUCTING BASES OF THE ALGEBRA OF PRIMITIVE- RECURSIVE FUNCTIONS

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In this paper we study the algebra of primitive-recursive functions $\mathfrak{A}_{\text{pr}} = \langle A_{\text{pr}}; +, *, \vdash \rangle$, where A_{pr} is the set of all unary primitive-recursive functions, and $+$, $*$, and \vdash are the operations of addition, superposition, and iteration of functions. In paper ⁽¹⁾ it was shown that the functions $\lambda_0(x) = x + 1$ and $q(x) = x - [\sqrt{x}]^2$ form a basis of the algebra \mathfrak{A}_{pr} . In other words, from the functions $\lambda_0(x)$ and $q(x)$, by means of the operations $+$, $*$, \vdash , one can obtain any primitive-recursive function. In paper ⁽²⁾ this result is generalized as follows: $\lambda_0(x)$ and $Q(x)$ are a basis for the algebra \mathfrak{A}_{pr} , where $Q(x)$ denotes the distance from x to the nearest number on the left in a certain sequence of numbers, which in that paper is called an arithmetic progression of the second order. In the present paper this result is generalized to arithmetic progressions of the k -th order, where $k \geq 2$. In particular, instead of the function $q(x)$ one may take the function $Q(x) = x[\sqrt[k]{x}]^n$. For every natural number $k \geq 2$, consider the class of unary primitive-recursive functions P_k , defined as follows. A primitive-recursive function $A(x)$ belongs to the class P_k if: 1) $A(0) = 0$; 2) $A(x)$ is strictly increasing; 3) $A(x)$ is an arithmetic progression of the k -th order, i.e., the finite differences of the k -th order of the sequence

$$A(0), A(1), A(2), \dots, A(n), \dots \quad (1)$$

are constant. In what follows, functions from P_k will be called, for short, **arithmetic progressions of the k -th order**.

Denote by $Q_A(x)$ the distance from x to the nearest number on the left of the form $A(y)$. Note that $Q_A(x)$ is a primitive-recursive function.

Main theorem. *The functions $\lambda_0(x) = x + 1$ and $Q_A(x)$ form a basis of the algebra \mathfrak{A}_{pr} .*

Concerning arithmetic progressions we shall give, without proof, the following lemma.

Lemma 1. *For every arithmetic progression of the k -th order $A(x)$ there exists a natural number i such that*

$$A(x+i) - A(i) = B(x) + x,$$

where $B(x)$ is a certain polynomial of degree k with positive rational coefficients, taking natural values for natural x .

Note that $B(x)$ will also be an arithmetic progression of the k -th order.

For every function $\varphi(x)$ consider the following function:

$$\alpha_\varphi(x) = \begin{cases} 1, & \text{if } x = \varphi(y) \text{ for some } y, \\ 0 & \text{in all other cases.} \end{cases}$$

Consider a strictly increasing function $F(x)$ such that $F(0) = 0$. For $F(x)$ we introduce the following functions:

$$f(x) = F(x) + x;$$

$$T_F(0) = 0, \quad T_F(x+1) = T_F(x) + \alpha_F(x+1);$$

$$\beta_f(x) = L[x+1 + \alpha_f(x+2)].$$

Lemma 2. $\beta_f(x) = x + T_F(x)$.

Proof. First we prove that

$$\alpha_f(x + T_F(x) + 2) = \alpha_F(x + 1). \quad (2)$$

Let $x = F(y) + k$, where $0 \leq k < F(y+1) - F(y)$. Then $j = x + T_F(x) + 2 = F(y) + k + y + 2$, since $T_F(F(y)) = y$. Hence $F(y) + y < F(y) + y + 2 \leq j < F(y+1) + (y+1) + 1$. Therefore $j = F(z) + z$ if and only if $z = y + 1$, and hence $x + T_F(x) + 2 = f(z)$ if and only if $x + 1 = F(z)$. Thus (2) is proved.

We now prove that $\beta_f(x) = x + T_F(x)$. If $x = 0$, then $\beta_f(0) = 0$ and $0 + T_F(0) = 0$. Suppose it has already been proved that, for some x , $\beta_f(x) = x + T_F(x)$. Then

$$\beta_f(x+1) = x + T_F(x) + 1 + \alpha_f(x + T_F(x) + 2) = x + T_F(x) + 1 + \alpha_F(x+1) = (x+1) + T_F(x+1).$$

Lemma 2 is proved.

Lemma 3. $\alpha_F(x) = \alpha_f(\beta_f(x))$.

Proof. The functions $F(x)$ and $\beta_f(x)$ are strictly increasing, and if $x = F(y)$, then $\beta_f(x) = F(y) + y = f(y)$. Lemma 3 is proved.

Consider two more series of functions

$$g_n(x) = \begin{cases} 1, & \text{if } x = n, \\ 0 & \text{in all other cases;} \end{cases}$$

$$\varphi_n(x) = \begin{cases} 1, & \text{if } x = ny, \\ 0 & \text{in all other cases,} \end{cases}$$

and, for a natural number $a \neq 0$, the formula

$$\gamma_a(x) = g_2[\alpha_F(x) + \varphi_a(\beta_f(x) + (a-1)x)].$$

Lemma 4. $\gamma_a(x) = \alpha_K(x)$, where $K(x) = F(ax)$.

Proof. $\gamma_a(x) = 1$ if and only if $x = F(y)$ for some y and $\beta_f(x) + (a-1)x$ is divisible by a ; this is possible only in the case when $y = az$ for some z . Thus $\gamma_a(x) = 1$ if and only if $x = F(az) = K(z)$ for some z . Lemma 4 is proved.

Let

$$M(x) = a_0x^k + a_1x^{k-1} + \dots + a_{k-1}(x),$$

where all a_i are natural and $k \geq 2$. Put

$$b = M(1),$$

$$L(x) = \frac{M(x+2) - M(x+1) - b}{x+1},$$

$$S(0) = 0, \quad S(x+1) = S(x) + L(x) + \text{sg } x$$

and choose the number d so that $b + d \neq M(y)$ for any y .

Consider the formula

$$\delta(x) = L[x + T_M(x) + b\alpha_M(x)] + bg_0(x) + dg_1(x).$$

Lemma 5. $\delta(x) = M(y+1)$ if and only if $x = S(y)$, and also $\alpha_S(x) = \alpha_M(\delta(x))$.

Proof. First, $\delta(0) = b = M(1)$. Further, we have $S(1) = L(0) + 1 = M(2) - 2b + 1 \neq 1$ and $\delta(1) = b + d \neq M(z)$ for any

some z . Continuing the computation, we shall have

$$\delta(2) = 2b + 1, \dots, \delta(S(1)) = \delta(M(2) - 2b + 1) = M(2).$$

Suppose that for $x = S(y + 1)$ $\delta(x) = M(y + 2)$. Then

$$\begin{aligned} \delta(S(y + 2)) &= \delta(S(y + 1) + L(y + 1)) = \\ &= M(y + 2) + (y + 2)L(y + 1) + b = M(y + 3). \end{aligned}$$

Lemma 5 is proved.

Lemma 6. $\alpha_\lambda(x) = \alpha_S(x + \text{sg } x)$, where

$$\lambda(0) = 0, \quad \lambda(x + 1) = \lambda(x) + L(x).$$

Proof. We have $\lambda(x) + \text{sg } x = S(x)$. Note that, since $L(x)$ is a polynomial with natural coefficients and $L(0) \neq 0$, $\lambda(x)$ is an arithmetic progression of order $k - 1$.

Theorem 1. If $A(x)$ is an arithmetic progression of order k and $k \geq 2$, then from the functions $\lambda_0(x) = x + 1$, $\alpha_A(x)$, by means of $+$, $*$, and $\lfloor \cdot \rfloor$, one can obtain the functions $x + \lfloor \sqrt{x} \rfloor$, x^2 , $B(px + q)$, and $A(px + q + r)$ for certain natural $p \neq 0$, q , and r .

Proof. 1) Let $A(x)$ be an arithmetic progression of second order. By Lemma 1 we find i and $B(x)$. It is easy to note that $\alpha_{B(x)+x}(y) = \alpha_A(y + A(i))$. By Lemmas 2 and 3, we obtain $\alpha_B(x)$. Let us find a number $a \neq 0$ such that $K(x) = B(ax) = dx^2 + ex$, where d and e are natural numbers. By Lemma 4 we obtain $\alpha_K(x)$. Then $\alpha_M(x) = \alpha_K(x + K(1))$, where $M(x) = dx^2 + (2d + e)x$. Applying Lemmas 2 and 3 several times, we obtain $\alpha_{K_1}(x)$, where $K_1(x) = dx^2 + dx$. It is easy to note that $\alpha_{K_1}(dx) = \alpha_{K_2}(x)$, where $K_2(x) = x^2 + x$. By Lemma 2 we obtain $x + \lfloor \sqrt{x} \rfloor$ and $\alpha_{K_3}(x)$, where $K_3(x) = x^2$. As is known (see, for example, (3), pp. 78-79),

$$\lfloor x + 1 + 2\alpha_{K_3}(x + 4) \rfloor = x + 2\lfloor \sqrt{x} \rfloor,$$

$$\lfloor x + 2\lfloor \sqrt{x} \rfloor + 1 \rfloor = x^2.$$

It remains for us to obtain the functions $B(px + q)$ and $A(px + q + r)$. Having obtained x^2 , one can obtain $K(x)$, i.e. $B(ax)$. But

$$A(ax + i) = B(ax) + ax + A(i).$$

2) Let $A(x)$ be an arithmetic progression of order $k + 1$, and suppose that for all arithmetic progressions of order k the theorem has been proved. As in the first case, we find $\alpha_B(x)$. Let us find a number $a \neq 0$ such that the function $K(x) = B(ax)$ is a polynomial with natural coefficients. Then $\alpha_{M(x)+x}(y) = \alpha_K(x + K(1))$, where $M(x)$ is some polynomial with natural coefficients. Applying Lemmas 2-6, we obtain $\alpha_\lambda(x)$, where $\lambda(x)$ is an arithmetic progression of order k . By the induction hypothesis one can obtain $x + [\sqrt{x}]$, x^2 , and $\lambda(px + q + r)$ for certain $p \neq 0$, q , and r . We have

$$\lambda(px + q + r) + \text{sg}(px + q + r) = S(px + q + r),$$

$$\delta(S(px + q + r)) = M(px + q_1),$$

$$M(px + q_1) + (px + q_1) + K(1) = K(px + q_2),$$

$$K(px + q_2) = B(p_1x + q_3),$$

$$B(p_1x + q_3) + (p_1x + q_3) + A(i) = A(p_1x + q_3 + i).$$

Theorem 1 is proved.

Remark. In the proof of the theorem the functions $g_n(x)$, $\varphi_n(x)$, which are easily obtained from the functions $x + 1$ and $\text{sg } x$, are used. The function $\text{sg } x = d_A(a + \text{sg } x)$, where a is such a number that $a = A(y)$ for some y , but $a + 1 \neq A(z)$ for any z . The functions $mx + n$ and $\text{sg } x$ are easily obtained from $x + 1$.

Now we shall prove the main theorem.

Theorem 2. The functions $\lambda_0(x) = x + 1$ and $Q_A(x)$ form a basis of the algebra \mathfrak{A}_{pr} .

Proof. Let

$$P(x, y) = A(px + py + q + i) + (x - y).$$

Then

$$P(x, y) = B(px + py) + q + (p + 1)x + (p - 1)y + q + A(i).$$

It is not difficult to note that $P(x, y) < A(px + py + q + i + 1)$. We have

$$\overline{\text{sg}} x = \text{sg } Q_A(a + \text{sg } x), \quad a_A(x) = \overline{\text{sg}} Q_A(x),$$

where a is a number such that $a \neq A(y)$ for any y , but $a + 1 = A(z)$ for some z . By Theorem 1 we obtain the function $B(px + q)$, and hence also $P(x, y)$. Then

$$\varepsilon(x, y) = Q_A(P(x, y)) = x - y, \quad \text{if } x \geq y.$$

By Theorem 1 we also obtain the functions $x + [\sqrt{x}]$ and x^2 . Then

$$\varepsilon(x + [\sqrt{x}], x) = [\sqrt{x}], \quad \varepsilon(x, [\sqrt{x}]^2) = q(x).$$

The proof of Theorem 2 is complete, since, by R. Robinson's theorem, $x + 1$ and $q(x)$ are a basis of the algebra \mathfrak{A}_{pr} .

Theorem 3. The functions $\lambda_0(x) = x + 1$ and $nQ_A(x)$ ($n \neq 0$) form a basis of the algebra \mathfrak{A}_{pr} .

Proof. $a_A(x)$ is obtained as in Theorem 2. Instead of the function $\varepsilon(x, y)$ one can obtain $n\varepsilon(x, y)$. The rest of the proof repeats Theorem 2 verbatim from (2).

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REFERENCES

- ¹ R. M. Robinson, *Bull. Am. Math. Soc.*, **53**, 925 (1947).
- ² I. A. Lavrov, E. A. Palyutin, Seventh All-Union Colloquium on General Algebra, Summaries of Communications and Reports, Kishinev, 1965.
- ³ R. Peter, *Recursive Functions*, Moscow, 1954.

Note: Figure translations are in progress. See original paper for figures.

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