ON A WEAK SOLUTION OF THE CAUCHY PROBLEM FOR A FIRST-ORDER EQUATION WITH TWO INDEPENDENT VARIABLES

1967

SovietRxiv

Abstract

Full Text

UDC 517.945

MATHEMATICS

N. N. KUZNETSOV

ON A WEAK SOLUTION OF THE CAUCHY PROBLEM FOR A FIRST-ORDER EQUATION WITH TWO INDEPENDENT VARIABLES

(Presented by Academician A. N. Tikhonov on 21 I 1967)

1. In this article we investigate the uniqueness of a weak solution of the equation

$$du/dt + \partial\varphi(u,t,x)/\partial x = 0, \tag{1}$$

satisfying the initial condition

$$u(0,x) = u^0(x). \tag{2}$$

Assuming in what follows that the function $u^0(x)$ is measurable and bounded on the entire axis, by a weak solution of the Cauchy problem (1)–(2) in the strip

$$S = \{0 \le t \le T,\ -\infty < x < \infty\}$$

we shall mean a bounded measurable function satisfying equation (1) and condition (2) in the sense of the theory of distributions (^{1}).

General results concerning the uniqueness of the solution of this problem, and due to O. A. Oleinik (^{1}), were obtained only for equations with a convex (in u) function φ. For equations not satisfying the convexity condition, O. A. Oleinik found a generalization of the stability condition in a certain rather narrow class of functions $\mathcal{K}$ and proved uniqueness of the solution in this class (^{2}). The class $\mathcal{K}$ consists of functions continuously differentiable everywhere in S, except for a finite number of smooth curves, on each of which, everywhere (except possibly for a finite number of points), one-sided limiting values exist on each side of the line of discontinuity. Moreover, at all points of the line of discontinuity (with the possible exception of a finite number of them) the condition is fulfilled:

$$\begin{gathered}[\varphi(u(t,x-0),t,x) - \varphi(w,t,x)]/[u(t,x-0) - w] \ge \\ \ge [\varphi(u(t,x-0),t,x) - \varphi(u(t,x+0),t,x)]/[u(t,x-0) - u(t,x+0)].\end{gathered} \tag{3}$$

(w is any value intermediate between $u(t, x-0)$ and $u(t, x+0)$.)

In the present article the uniqueness theorem is extended to a broad class of weak solutions. Along the way, the Cauchy problem for a linear equation of the form (1) is considered.

2. We shall assume that the functions $\varphi(u, t, x)$, $\varphi_u(u, t, x)$ are defined and continuous for $t, x \in S$ and all values of u. We shall also assume that, for fixed u, t, $\varphi_u/x \to 0$ as $|x| \to \infty$. Denote by $\mathcal{H}$ the set of functions $u(t, x)$, measurable and bounded in S, which have, almost everywhere on every curve that is the graph of a Lipschitz-continuous function $x = \psi(t)$ $(0 \le t \le T)$, limiting values

$$u(t, \psi(t) \pm 0) = u^{\pm}(t, \psi).$$

Obviously, $\mathcal{H}$ contains, in particular, functions that have locally bounded variation for almost every t, functions continuous in x, etc. Introduce in $\mathcal{H}$ the subset $\mathcal{H}_s$ of functions satisfying, almost everywhere on every Lipschitz-continuous curve $x = \psi(t)$, the condition

$$(u^+ - u^-)\{\varphi(\alpha u^- + \beta u^+, t, \psi) - (\alpha\varphi^+ + \beta\varphi^-)\} \ge 0 \tag{4}$$

for any nonnegative $\alpha, \beta,\ \alpha + \beta = 1$. It is easy to verify that in the class $\mathcal{H}$ conditions (4) and (3) coincide.

Let us note some properties of weak solutions.

It follows directly from the definition that, whatever the smooth finite function $f(x)$, the function

$$\int_{-\infty}^{\infty} f(x) u(t, x)\, dx$$

is absolutely continuous in t (up to equivalence). Hence it follows that the same is true for the functions

$$u^h(t, x) = \frac{1}{h}\int_x^{x+h} u(t, \xi)\, d\xi, \qquad \varphi^h(t, x) = \frac{1}{h}\int_x^{x+h} \varphi(u(t, \xi), t, \xi)\, d\xi.$$

If $\psi(t)$ is an arbitrary Lipschitz-continuous function and $u \in \mathcal{H}$ is a weak solution, then, since the functions $u^h(t, \psi(t))$, according to the preceding, are continuous and, as $h \to +0$, converge almost everywhere to $u^+(t, \psi(t))$, the function $u^+(t, \psi(t))$ is measurable and bounded. The same is true for the functions $u^-(t, \psi(t))$, $\varphi(u^{\pm}, t, \psi)$. Directly from the definition of a weak solution it follows that almost everywhere on the curve $x = \psi(t)$

$$\varphi(u^+, t, \psi) - \varphi(u^-, t, \psi) = \psi'(t)(u^+ - u^-). \tag{5}$$

Consider the domain

$$G_{t_0} = \{(t,x) \mid 0 \le t \le t_0 \le T,\ \psi_1(t) \le x \le \psi_2(t)\},$$

where ψ_1, ψ_2 are Lipschitz-continuous. We shall show that for every weak solution of problem (1)–(2) from the class $\mathscr{H}$, and for almost all $t_0 \in [0,T]$,

$$\int_{\psi_1(t_0)}^{\psi_2(t_0)} u(t_0,x)\,dx - \int_{\psi_1(0)}^{\psi_2(0)} u^0(x)\,dx = \int_0^{t_0} (u^+\psi_2' - \varphi^+)_{x=\psi_2}\,dt - \int_0^{t_0} (u^+\psi_1' - \varphi^+)_{x=\psi_1}\,dt. \tag{6}$$

Indeed, by virtue of the absolute continuity of the functions u^h, we have for almost all t

$$\partial u^h(t,x)/\partial t + \partial \varphi^h(t,x)/\partial x = 0.$$

Integrating this identity (in the sense of Lebesgue) over the domain G_{t_0} and applying Fubini' s theorem, we obtain for almost all t_0

$$\int_{\psi_1(t_0)}^{\psi_2(t_0)} u^h(t_0,x)\,dx - \int_{\psi_1(0)}^{\psi_2(0)} u^{0h}(x)\,dx = \int_0^{t_0} (u^h\psi_2' - \varphi^h)_{x=\psi_2}\,dt - \int_0^{t_0} (u^h\psi_1' - \varphi^h)_{x=\psi_1}\,dt. \tag{7}$$

Since from the absolute continuity of the Lebesgue integral it follows that for any x_1, x_2

$$\int_{x_1}^{x_2} [u^h(t,x) - u(t,x)]\,dx \to 0$$

as $h \to 0$, passing to the limit in (7), we obtain (6), as was required to prove. We note that, according to relation (5), the right limiting values in (6) may be replaced by the left ones.

3. Let us prove the uniqueness of the weak solution of problem (1)–(2) in the class of functions $\mathscr{H}_s$.

Let $u_1, u_2 \in \mathscr{H}_s$ be two weak solutions of equation (1). Put

$$f(t,x) = \begin{cases} [\varphi(u_1,t,x) - \varphi(u_2,t,x)]/(u_1 - u_2), & u_1 \ne u_2, \\ \varphi_u(u_1,t,x), & u_1 = u_2. \end{cases}$$

Denote

$$f_N = \operatorname*{vrai\,max}_{0<t<T,\ |x|<N} |f(t,x)|$$

and

$$S_N = \{(t,x) \mid 0 \le t \le T,\ |x| \le N - t f_N\}.$$

From our assumptions it follows that $N^{-1} f_N \to 0$ as $N \to \infty$,

so that S_N contains any preassigned finite subdomain of the strip S, provided only that N is sufficiently large.

A Lipschitz-continuous function $x = \psi(t)$, defined for $0 \le t \le T$, will be called a solution of the equation

$$dx/dt = f(t,x), \tag{8}$$

if it satisfies this equation in the sense of Filippov (^{3}), which in our class of functions obviously means that the inequalities

$$\min(f^+(t,\psi), f^-(t,\psi)) \le \psi'(t) \le \max(f^+(t,\psi), f^-(t,\psi))$$

hold for almost all values of t. In (^{3}) it is proved that through every point (t_0, x_0) there exists a solution passing through this point. If $(t_0, x_0) \in S_N$, then this solution lies entirely in S_N.

Lemma 1. If $\psi(t)$ is a solution of (8) passing through the point $(t_0, x_0) \in S_N$, then

$$\operatorname{mes}\{t \mid f^-(t,\psi) < \psi'(t) < f^+(t,\psi)\} = 0.$$

This lemma is a consequence of inequality (4), which is satisfied on the line $x = \psi(t)$ by both solutions. We omit the detailed proof.

Denote by $\psi^*(t)$ the upper solution passing through the point $(t_0, x_0) \in S_N$. The existence of such a solution was proved in Theorem 6 of (^{3}).

Lemma 2. $\operatorname{mes}\{t,\ t \le t_0 \mid d\psi^*/dt > f^+(t,\psi^*)\} = 0.$

Indeed, introduce the function

$$F(t,x) = \begin{cases} -2f_N, & x \le \psi^*(t), \\ f(t,x), & x > \psi^*(t), \end{cases}$$

and let $\psi_1^*(t)$ be the upper solution, passing through the point (t_0, x_0), of the equation

$$dx/dt = F(t,x).$$

Obviously, for $t \le t_0$ it lies in the region $x \ge \psi^*(t)$ and therefore is a solution of equation (8). Since $\psi^*(t)$ is the upper solution, it follows that almost everywhere for $t \leq t_0$, $\psi_1^*(t) = \psi^*(t)$. Since $\max(F^-(t, \psi^*), F^+(t, \psi^*)) = f^+(t, \psi^*)$, it follows that almost everywhere for $t \leq t_0$, $d\psi^*/dt \leq f^+(t, \psi^*)$, as was required to prove.

Lemma 3. On the line $x = \psi^*(t)$, for almost all $t \leq t_0$,

$$(u_1^+ - u_2^+)\, d\psi^*/dt = \varphi(u_1^+, t, \psi^*) - \varphi(u_2^+, t, \psi^*). \tag{9}$$

Indeed, by the preceding lemma, it is enough to prove (9) only on the set M where $\psi^{*\prime} < f^+(t, \psi^*)$. From Lemma 1 it follows that almost everywhere on M, $\psi^{*\prime} \leq f^-(t, \psi^*)$. Since $\psi^{*\prime} \geq \min(f^-, f^+)$, it follows that almost everywhere on M, $\psi^{*\prime} = f^-(t, \psi^*)$, and (9) follows from (5). Lemma 3 is proved.

Let $(t_0, x_1), (t_0, x_2) \in S_N$, $x_1 < x_2$. Let $\psi_1(t)$ and $\psi_2(t)$ be the upper solutions of equation (8), passing through the points (t_0, x_1) and (t_0, x_2), respectively. Obviously, $\psi_1 \leq \psi_2$. Then from Lemma 3 and identity (6) we obtain, for almost every $t_0 \in [0, T]$,

$$\int_{x_1}^{x_2} [u_1(t_0, x) - u_2(t_0, x)]\, dx = \int_{\psi_1(0)}^{\psi_2(0)} [u_1^0(x) - u_2^0(x)]\, dx,$$

where $u_i^0(x)$ are the initial functions corresponding to the solutions under consideration. If $u_1^0(x) = u_2^0(x)$, then almost everywhere in S_N, and consequently in

$$S = \bigcup_N S_N,$$

$u_1 = u_2$. The uniqueness theorem is proved.

4. Consider the case of a linear equation: $\varphi = f(t, x)u$. We shall assume that $f(t, x)$ is defined and measurable in S, and that $f_N = o(N)$ as $N \to \infty$, where

$$f_N = \operatorname*{vrai\,max}_{0 \leq t \leq T,\ |x| \leq N} |f(t, x)|.$$

Assume, further, that: a) for every smooth-

of any finite function $g(x)$ the function

$$\int_{-\infty}^{\infty} f(t, x) g(x)\, dx$$

is continuous in t; b) almost everywhere on every Lipschitz-continuous curve $x = \psi(t)$ there exist limiting values $f^\pm(t, \psi)$; c) for the function $f(t, x)$ the assertion of Lemma 1 is valid.

Then the weak solution $u \in \mathcal{H}$ of the Cauchy problem (1)–(2) with $u^0(x) = 0$ is equal to zero everywhere in S.

For the proof it suffices to replace relations (5) and (6), respectively, by

$$f^+(t,\psi)u^+(t,\psi) - f^-(t,\psi)u^-(t,\psi) = \psi'(t)(u^+ - u^-),$$

$$\int_{\psi_1(t_0)}^{\psi_2(t_0)} u(t_0,x)\,dx - \int_{\psi_1(0)}^{\psi_2(0)} u^0(x)\,dx = \int_0^{t_0} \left(u^+\psi_2' - f^+u^+\right)_{x=\psi_2} dt - \int_0^{t_0} \left(u^+\psi_1' - f^+u^+\right)_{x=\psi_1} dt$$

and to repeat the arguments of the preceding item following Lemma 1.

Moscow State University
named after M. V. Lomonosov

Received
21 I 1967

REFERENCES

1. O. A. Oleinik, *UMN*, **12**, No. 3, 3 (1957).
2. O. A. Oleinik, *UMN*, **14**, No. 2, 165 (1959).
3. A. F. Filippov, *Matem. sborn.*, **51**, No. 1, 99 (1960).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.