The solution of singular Cauchy problems in basis series

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Abstract

The paper investigates a singular Cauchy problem for the generalized wave equation

$$z_{xx} = z_{ss} + \frac{a}{s} z_s + b^2 z, \quad z(x,0) = \tau(x), \quad \tau_s(x,0) = 0. \tag{1}$$

Using the integral representation of its solution, the author constructs basis series expansions of two types for $z(x, \lambda s; a, b)$, $\lambda = \text{const}$:

$$z(x, \lambda s; a_2, b) = \sum_{n=0}^{\infty} A_n(\lambda, s) D_x^{2n} z(x, s; a_1 + 4n, 0), \tag{2a}$$

$$z(x, \lambda s; a_2, b) = \sum_{n=0}^{\infty} \overline{A}_n(\lambda, s) D_x^{2n} z(x, s; a_1 + 2n, 0). \tag{2b}$$

The cases where the confluent Horn hypergeometric functions $\Xi_2(\sigma, \beta, \gamma; x, y)$, included in $A_n(\lambda, s)$ and $\overline{A}_n(\lambda, s)$, degenerate into Jacobi and Gegenbauer polynomials are considered, as well as the case of Ξ degenerating into Bessel functions, where (2a) yields an addition theorem for $z(x, s; a, b)$ with respect to the parameter b. The inhomogeneous singular Cauchy problem

$$u_{xx} = u_{ss} + \frac{a}{s} u_s + \frac{c}{s^2} u - \frac{c}{s^2} \tau(x), \quad u(x,0) = \tau(x), \tag{3}$$
$$u_s(x,0) = 0.$$

is solved in similar basis series. Confluent initial problems arising from (1) and (3) under the simultaneous growth of parameters a, c and the argument s are also investigated. The results obtained from considering problems (1) and (3) are used to construct algorithms for the effective solution of a more complex singular Cauchy problem associated with the S. A. Chaplygin equation

$$u_{xx} = u_{ss} + B(s) u_s, \quad u(x,0) = \tau(x), \quad u_s(x,0) = 0. \tag{4}$$

Seeking $u(x, s; B)$ in the form of basis series similar to (2a) and (2b), the author arrives at a recurrence sequence of second-order ordinary differential equations for the coefficients $A_n(s)$ of these series, which is then solved explicitly (in quadratures). Using the found orthogonal expansions in Jacobi and Gegenbauer polynomials, and in confluent cases in Laguerre and Hermite polynomials, resolving operators for problems (3) and (4) are constructed in integral form, as well as transformation operators mapping $z(x, s; a, 0)$ into $u(x, s; a, c)$ and $u(x, s; B)$. The obtained results are applied to the derivation of functional relations for hypergeometric series with one and two arguments, Lommel functions, Struve functions, and other higher transcendental functions, as well as to the generalization of self-similar integrals of the Euler-Poisson equation to the case (4). Along with this, the paper obtains the Green-Hadamard functions for two singular Tricomi problems for an equation with two lines of degeneracy in the form of Horn hypergeometric series $H_2(\alpha, \beta, \gamma; \delta; \varepsilon; x, y)$, which make it possible to reduce the solutions of these problems to quadratures. Bibliography: 14 items.

Full Text

Preamble

This section explores the relationships between solutions to various partial differential equations, specifically focusing on the properties of the function $z(x, s; a, b)$ and its generalizations. We consider the differential operator $D_x = \frac{\partial}{\partial x}$ and its applications to sequences of functions $z(x, s; a + 2n, 0)$ and $D_x^n z(x, s; a + 4n, 0)$ for $n = 0, 1, 2, ...$ Building upon the classical work of S. A. Chaplygin, we analyze the behavior of $z(x, s; a, b)$ and its transformations into $u[x, s; B(s)]$ and $u(x, s; a, c)$. These relationships allow for the representation of solutions to complex boundary value problems through series expansions and integral operators.

§ 1. Basic Relations

Following the methodology established in [1], we consider the function $z(x, s; a, b)$ which satisfies the following second-order partial differential equation:

$$z_{xx} = z_{ss} + \frac{a}{s} z_s + b^2 z, \quad z(x, 0) = \tau(x), \quad z_s(x, 0) = 0 \tag{1.1}$$

where $a = 2\nu > 0$. The solution can be expressed in the integral form:

$$z(x, s; a, b) = \int_{-1}^{1} \tau(x + \xi s)(1 - \xi^2)^{\nu - 1/2} T(\xi, s) d\xi \tag{1.2a}$$

In the case where $b = 0$, the solution for the parameter $a + 2n$ is denoted as $z(x, s; a+2n, 0)$. Using the properties of the operator D_x^n, we derive the following relationship for the n-th derivative:

$$s^n D_x^n z(x, s; a + 2n, 0) = A_n \int_{-1}^{1} \tau(\xi) d\xi \tag{1.3}$$

where the coefficients A_n are determined by the Gamma function $\Gamma(\nu)$ and the specific indexing of the parameter a.

By substituting the integral representation (1.2a) into the governing equation, we obtain a series expansion for the general case $b \neq 0$:

$$z(x, \lambda s; a_2, b) = \sum_{n=0}^{\infty} g_n(\nu) s^{2n} D_x^n z(x, s; a_1 + 2n, 0) \tag{1.9a}$$

where the coefficients $g_n(\nu)$ involve hypergeometric functions of the form:

$$g_n(\nu) = \frac{k^2 \lambda^2 s^2}{2^{2n} n! (\nu + 1)_n} F(-n, \nu_1 + 1, \nu_2 + 1; k^2 \lambda^2 s^2) \tag{1.9b}$$

This formulation allows us to relate solutions with different parameters a_1 and a_2 through the action of the spatial derivative operator.

§ 2. Integral Representations and Transformations

For the case where $b = 0$, the relationship between the initial data $\tau(x)$ and the solution $z(x, s; a, 0)$ can be inverted. Specifically, we can express $\tau(x)$ as a series:

$$\tau(x) = \sum_{n=0}^{\infty} g_n(\nu) s^{2n} D_x^n z(x, s; a + 4n, 0) \tag{1.10a}$$

Alternatively, using the operator $W = \sqrt{s^2 D_x^2 - b^2}$, we can write:

$$\tau(x) = W z(x, s; a, b) \tag{1.10c}$$

These expressions demonstrate that the solution at any $s > 0$ contains sufficient information to reconstruct the initial state $\tau(x)$ via differential operators.

When considering the transformation to the function $u(x, s; a, c)$, we utilize the Gauss hypergeometric function $F(-n, p+n, q; x)$. For $a_1 > a_2 > 0$, the following integral relation holds:

$$s^{2n} D_x^{2n} z(x, s; a_1 + 4n, 0) = \delta_n \int_{-1}^{1} (1 - \xi^2)^{\nu_1 - 1/2} G_n(\nu_1, \nu_2 + 1; \xi^2) z(x, \xi s; 0, 0) d\xi \tag{1.12}$$

where δ_n is a normalization constant involving $\Gamma(\nu_1 + 1)$ and $\Gamma(\nu_1 - \nu_2 + n)$. In the limit where $a_2 = 0$ and $b = 0$, the expression simplifies to:

$$z(x, \lambda s; 0, 0) = \sum_{n=0}^{\infty} \gamma_n(\nu) s^{2n} D_x^n z(x, s; a + 4n, 0) \tag{1.13a}$$

This allows for a direct connection between the symmetric mean of the initial data $2z(x, \lambda s; 0, 0) = \tau(x + \lambda s) + \tau(x - \lambda s)$ and the higher-order parameter solutions.

§ 3. Generalized Hypergeometric Representations

We further generalize these results by setting $\lambda = 1$ and $a_1 = a_2 = a$ in the previous relations. The function $z(x, s; a, b)$ can be expanded as:

$$z(x, s; a, b) = \sum_{n=0}^{\infty} A_n s^{2n} D_x^n z(x, s; a + 2n, 0) \tag{1.15a}$$

where the coefficients A_n are related to the Bessel function $J_{\nu+2n}(bs)$ as follows:

$$A_n = \frac{(-1)^n \Gamma(\nu + n)}{n!\Gamma(\nu + 2n)} \left(\frac{bs}{2}\right)^n J_{\nu+2n}(bs) \tag{1.15b}$$

This representation is particularly useful for analyzing the asymptotic behavior of the solution as the parameter b varies.

For the function $u(x, s; a, c)$, we obtain a similar expansion:

$$u(x, \lambda s; a_2, c) = \sum_{n=0}^{\infty} A_n s^{2n} D_x^n z(x, s; a_1 + 4n, 0) \tag{1.12a}$$

The coefficients here involve the generalized hypergeometric function ${}_3F_2$:

$$A_n = g_n(\nu_1)_3F_2(-n, \nu_1 + n, 1; p_2 + 1, q_2 + 1; \lambda^2) \tag{1.12b}$$

These results unify several known special cases in the theory of singular partial differential equations and provide a systematic way to construct solutions using known basis functions.

§ 4. Asymptotic Limits and Connections to Heat Equations

By performing a change of variables $s = 2\sqrt{\epsilon} s_1$ and $a = 2\epsilon - 1$, and taking the limit as $\epsilon \to \infty$, the equation (1.1) transforms into a parabolic type:

$$\lim_{\epsilon \to \infty} z(x, 2\sqrt{\epsilon} s_1; a, b) = w(x, s_1; b) \tag{4.1}$$

where w satisfies the heat-like equation $w_{s_1} = w_{xx} - b^2 w$. Under this limit, the differential operators transform as:

$$\lim_{\epsilon \to \infty} D_s^n z = D_{s_1}^n w \tag{4.3}$$

This allows us to map the properties of the hyperbolic equation (1.1) onto the parabolic domain. For instance, the expansion (1.9) becomes:

$$w(x, s_1; b) = \sum_{n=0}^{\infty} \frac{A_n s_1^n}{n!} D_x^n w(x, s_1; 0) \tag{4.4a}$$

which is a known result in the theory of heat kernels.

Furthermore, the integral representation (1.2a) in this limit yields:

$$w(x, s_1; 0) = \int_{-\infty}^{\infty} \tau(x+\xi) \frac{1}{\sqrt{4\pi s_1}} e^{-\xi^2/4s_1} d\xi \tag{4.6}$$

This confirms that our generalized framework correctly reduces to the classical Weierstrass transform for the heat equation. Similar limits applied to the $u(x, s; a, c)$ function lead to new representations for solutions of confluent hypergeometric differential equations.

§ 5. Applications to Chaplygin-type Equations

Finally, we apply these methods to the generalized Chaplygin equation:

$$u_{ss} + B(s)u_s = u_{xx}, \quad u(x, 0) = \tau(x) \tag{5.1a}$$

where $B(s) = \frac{d}{ds} \ln \sqrt{K(s)}$. Using the integral operator (5.2), we represent the solution as:

$$u(x, s; B) = \sum_{n=0}^{\infty} A_n(s) D_x^n z(x, s; a + 4n, 0) \tag{5.3a}$$

The coefficients $A_n(s)$ are determined by a system of recurrence relations:

$$A_{n+1} = -2 \int s A_n ds + \text{const} \tag{5.6}$$

This approach allows for the construction of solutions for various gas dynamics problems where the function $K(s)$ takes specific forms, such as $K(s) = s^m$ or more complex analytic expressions. The convergence of these series and their relationship to the fundamental solutions of the associated singular equations are established through the properties of the operator D_x^n.

§ 6. Examples and Special Cases

To illustrate the theory, we consider the case where $\tau(x)$ satisfies the harmonic condition $\tau''(x) + \kappa^2 \tau(x) = 0$. The solutions take the form of product functions:

$$z(x, s; a, b) = \tau(x) s^{\nu-1} J_\nu(b_0 s), \quad w(x, s; b) = \tau(x) e^{-b_0^2 s} \tag{6.1a, b}$$

where $b_0 = \sqrt{b^2 + \kappa^2}$. For polynomial initial data $\tau(x) = x^m$, the solutions are expressed via hypergeometric functions ${}_2F_1$ and ${}_3F_2$. These examples validate the general expansion formulas derived in the preceding sections and demonstrate their utility in practical calculations.

The results presented here extend the work of previous authors [8-12] by providing a unified operator-based approach to the study of singular partial differential equations. The use of the D_x operator and its associated series expansions offers a powerful tool for both theoretical analysis and the numerical approximation of solutions in mathematical physics.

Note: Figure translations are in progress. See original paper for figures.

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