ON AN ESTIMATE OF THE DEFECT OF A SPACE

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Abstract

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ON AN ESTIMATE OF THE DEFECT OF A SPACE

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Denote by $\operatorname{def} X$ the defect* of the space X, and by $P(X)$ the set of all points of the space X at which it is peripherally compact**. There are** (1)* examples of spaces X such that the set $X \setminus P(X)$ is compact and the defect $\operatorname{def} X$ is equal to $\dim(X \setminus P(X)) + 1$. We shall prove a theorem from which the inequality

$$\operatorname{def} X \leq \dim(X \setminus P(X)) + 1$$

follows, under the condition that the set $P(X)$ is open. All spaces considered will be metric spaces with a countable base.

Lemma 1. *If X is a totally bounded space and $\dim X \leq n$, then there exists a sequence $\varphi_0, \varphi_1, \dots$ of closed finite covers of the space X such that $\varphi_0 = \{X\}$, the diameter of each element of the cover φ_i is less than i^{-1}, and the multiplicity of the cover $\varphi_{i-1} \cup \varphi_i$ is less than $n+3$ $(i = 1, 2, \dots)$.*

Proof. We shall construct the covers φ_i by induction with respect to i. The cover φ_0 is defined and satisfies the following condition $(\mathfrak{M}_i)$ for $i = 0$:

$(\mathfrak{M}_i)$. The dimension of the common part of any family consisting of $r \leq n+2$ elements of the cover φ_i is less than $n - r + 2$.

Suppose now that $i > 0$ and that a closed finite cover $\varphi_{i-1} = \{F_1, \dots, F_l\}$ of the space X has been defined, satisfying condition $(\mathfrak{M}_{i-1})$. Decompose the space X into the sum of open sets $G_1, \dots, G_m$, the diameter of each of which is less than i^{-1}. By Menger' s theorem (2), there exists a closed cover $\varphi_i = \{F'_1, \dots, F'_m\}$ of the space X such that $F'_k \subseteq G_k$ for $k = 1, \dots, m$, and such that condition $(\mathfrak{M}_i)$ is satisfied together with the inequality

$$\dim(F_{j_0} \cap \dots \cap F_{j_r} \cap F'_{k_0} \cap \dots \cap F'_{k_s}) < \dim(F_{j_0} \cap \dots \cap F_{j_r}) - s$$

for any integers s in the interval $0 \leq s \leq d+1$, where

$$d = \dim(F_{j_0} \cap \dots \cap F_{j_r});$$

$0 \leqslant j_0 < \dots < j_r \leqslant l, \quad 0 \leqslant k_0 < \dots < k_s \leqslant m$. According to $(\mathfrak{N}_{i-1})$, we have $d \leqslant n - r$, if $r \leqslant n+1$. Take integers $r \geqslant 0$, $t \geqslant 0$ such that $r + t = n + 1$. Defining the number s by the formula

$$s = \begin{cases} t, & \text{if } t \leqslant d+1, \\ d+1, & \text{if } t > d+1, \end{cases}$$

we obtain $0 \leqslant s \leqslant d+1$, $s \leqslant t$. Moreover, $d - t \leqslant n - r - t = -1$. Then the common part of any family consisting of $r+1$ elements of the cover φ_{i-1} and $t+1$ elements of the cover φ_i has dimension

$$\dim(F_{j_0} \cap \dots \cap F_{j_r} \cap F'_{k_0} \cap \dots \cap F'_{k_t}) \leqslant$$

$$\leqslant \dim(F_{j_0} \cap \dots \cap F_{j_r} \cap F'_{k_0} \cap \dots \cap F'_{k_s}) \leqslant d - s = \begin{cases} d - t = -1, \\ d - (d+1) = -1. \end{cases}$$

Thus, the multiplicity of the cover $\varphi_{i-1} \cup \varphi_i$ is less than $n+3$; Lemma 1 is proved.

Let α, β be some families of sets. We shall denote by $\alpha \wedge \beta$ the family of all sets of the form $A \cap B$, where $A \in \alpha$, $B \in \beta$. The closure of a set A is denoted by $\operatorname{cl} A$. If α is a family of sets, then $\operatorname{cl} \alpha$ denotes the family of all sets $\operatorname{cl} A$, where $A \in \alpha$. Further, $|\alpha|$ is the union of all sets of the family α. Finally, let $I_0^{\aleph_0}$ denote the first left face of the Hilbert cube $I^{\aleph_0}$, i.e.,

$$I_0^{\aleph_0} = \{0\} \times [0,1] \times [0,1] \times \dots \subset [0,1] \times [0,1] \times [0,1] \times \dots = I^{\aleph_0}.$$

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* The least dimension of a remainder in compact extensions of a given space is called the **defect** of this space.

** A space X is called **peripherally compact at a point** $x \in X$ if x has arbitrarily small neighborhoods in X whose boundaries are compact.

*** The present paper to some extent supplements paper (^{1}), two assertions of which (namely, Corollaries 2.1 and 2.2) are not fully justified. We note that, in order to leave the assertion in Theorem 2 from (^{1}) unchanged, one must change the definition of the quantity $\operatorname{Com} X$, which is given in (^{1}). One possible approach to this problem is to change only the first step of the inductive definition of the inequality $\operatorname{Com} X \leq n$, with the induction beginning not from $n = -1$, but from $n = 0$. Thus, following Grot, the relation $\operatorname{Com} X \leq 0$ is defined as a condition equivalent to the equality $X = P(X)$. However, then the inequality $\operatorname{def} X \leq \operatorname{Com} X$ does not follow from the work of Vries. The theorem given below entails Corollaries 2.1 and 2.2 from (^{1}) for those spaces X for which the set $P(X)$ is open.

Lemma 2. *If X is a closed subset of a space Y, then there exists a continuous mapping $f : Y \to I^{\aleph_0}$ such that $f|X$ is a homeomorphism and*

$$f(X) \subseteq I_0^{\aleph_0}, \qquad f(Y \setminus X) \subseteq I^{\aleph_0} \setminus I_0^{\aleph_0}, \qquad \dim f(Y) \leqslant \dim X + 1.$$

Proof. We may immediately assume that $X \neq \emptyset$ and $Y \subseteq I^{\aleph_0}$. The mapping f will be an extension of the embedding $h : X \to I^{\aleph_0}$, $h(y) = y$ $(y \in X)$. This extension is trivial in the case when the set X is finite-dimensional. Thus, putting $n = \dim X$, consider a sequence of closed finite covers $\varphi_0, \varphi_1, \dots$ of the set X such that the assertion of Lemma 1 is satisfied. We may assume that all elements of the families φ_i are nonempty. Take $\gamma_{-1} = \{\emptyset\}$, $\gamma_0 = \{Y\}$, and note that the family γ_0 satisfies the following condition $(\mathfrak{N}_i)$ for $i = 0$:

$(\mathfrak{N}_i)$ The multiplicity of both the family $\operatorname{cl}\gamma_{i-1} \cup \operatorname{cl}\gamma_i$ and the family $\operatorname{cl}\gamma_i \cup \varphi_{i+1}$ is less than $n+3$.

Suppose that for some $i > 0$ a family γ_{i-1} has been defined such that $X \subseteq |\gamma_{i-1}|$, γ_{i-1} consists of open subsets of the space Y, and satisfies condition $(\mathfrak{N}_{i-1})$. We construct the family γ_i. Let $F \in \varphi_i$. Let F' be the union of all sets each of which is a common part, intersecting F, of a family of sets contained in the family $\operatorname{cl}\gamma_{i-1} \cup \varphi_i$ or in the family $\varphi_i \cup \varphi_{i+1}$. Since $F \subseteq |\gamma_{i-1}| \setminus F'$ and the diameter of F is less than i^{-1}, there exists an open subset G of the space Y such that $F \subseteq G \subseteq \operatorname{cl} G \subseteq |\gamma_{i-1}| \setminus F'$ and the diameter of G is less than i^{-1}. But from condition $(\mathfrak{N}_{i-1})$ and Lemma 1 it follows that the multiplicity of both families $\operatorname{cl}\gamma_{i-1} \cup \varphi_i$, $\varphi_i \cup \varphi_{i+1}$ is less than $n+3$. Consequently, less than $n+3$ also are the multiplicities of the families $\operatorname{cl}\gamma_{i-1} \cup \operatorname{cl}\varphi_i'$, $\operatorname{cl}\varphi_i' \cup \varphi_{i+1}$, where φ_i' is the family obtained from φ_i by replacing the set F by the set G. Replacing successively all sets of the family φ_i, we thus obtain a family γ_i such that $X \subseteq |\gamma_i| \subseteq \operatorname{cl}|\gamma_i| \subseteq |\gamma_{i-1}|$, γ_i consists of open subsets of the space Y and satisfies condition $(\mathfrak{N}_i)$. Moreover, the elements of the family γ_i intersect X and their diameters are less than i^{-1}.

It follows from this that $Y \setminus X$ is the sum of the sets $|\gamma_i| \setminus |\gamma_{i+1}|$ $(i = 0, 1, \dots)$. Put

$$\chi_i = \gamma_i \wedge \{\, Y \setminus \operatorname{cl}|\gamma_{i+2}|\,\}.$$

We have

$$|\gamma_i| \setminus |\gamma_{i+1}| \subset |\gamma_i| \setminus \operatorname{cl}|\gamma_{i+2}| = |\chi_i| \subset \operatorname{cl}|\chi_i| \subset Y \setminus |\gamma_{i+2}| \subset Y \setminus X.$$

Thus, $\chi = \chi_0 \cup \chi_1 \cup \cdots$ is an open countable covering of the set $Y \setminus X$; the closures of the elements of the covering χ lie in $Y \setminus X$, and their diameters tend to zero. Moreover, in the family χ each element intersects only a finite number of elements, and from condition $(\mathfrak{M}_i)$ it follows that the multiplicity of χ is less than $n+3$. Applying Kuratowski' s method (3) to the system χ and to its nerve, we obtain the required extension f of the embedding h. The set $f(Y \setminus X)$ is contained in the nerve of the covering χ, which is an infinite polyhedron lying in $I^{\aleph_0}/I_0^{\aleph_0}$ and having dimension less than $n+2$; Lemma 2 is proved.

Theorem. *For every space X the inequality*

$$\operatorname{def} X \leq \dim \operatorname{cl}(X \setminus P(X)) + 1$$

holds.

Proof. Denote by A the closure of the set $X \setminus P(X)$. Let $n = \dim A$. Take such a compact extension cX of the space X that the closure cA of the set A in cX has dimension n. From Lemma 2 there follows the existence of a continuous mapping

$$f : cX \to I_0^{\aleph_0}$$

such that $f|cA$ is a homeomorphism, $f(cA) \cap f(cX \setminus cA) = \emptyset$, and $\dim f(cX) \leq n+1$. There is a sequence of open finite coverings $\gamma_1, \gamma_2, \ldots$ of the image $f(cX)$, possessing the basis property* at each of its points and such that γ_{i+1} is star-refined into γ_i, and the multiplicity of γ_i is less than $n+3$ $(i = 1, 2, \ldots)$. Consider the open finite covering χ_i of the space X, consisting of the sets $f^{-1}(G) \cap X$, where $G \in \gamma_i$. Then the sequence of coverings $\chi_1, \chi_2, \ldots$ has the basis property at the points of the set A. Since $X \setminus A \subset P(X)$, there exists such a sequence $K_1, K_2, \ldots$ of open subsets of the space X that all boundaries $\operatorname{cl} K_i \setminus K_i$ are compact and every point of the set $X \setminus A$ lies in sets K_i of sufficiently small diameter. Put

$$\varkappa_i = \{K_i,\ X \setminus \operatorname{cl} K_i\}, \qquad \lambda_j = \chi_j \wedge \varkappa_1 \wedge \varkappa_2 \wedge \cdots \wedge \varkappa_j \quad (j = 1, 2, \ldots).$$

The set $X \setminus |\lambda_j|$ is the sum of the boundaries $\operatorname{cl} K_i \setminus K_i$, $i = 1, \ldots, j$. Hence all λ_j are borderings** of the space X, and the sequence $\lambda_1, \lambda_2, \ldots$ has the basis property at all points of this space. On the other hand, the bordering λ_{j+1} is star-refined into the bordering λ_j; the multiplicity of λ_j is less than $n+3$ $(j = 1, 2, \ldots)$. According to a theorem of Yu. M. Smirnov (4), the inequality

$$\operatorname{def} X \leq n + 1$$

is obtained.

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[*] A sequence of families $\gamma_1, \gamma_2, \ldots$ has the **basis property at a point** $x \in X$ if for every neighborhood $U \subseteq X$ of the point x there exist an integer $i > 0$ and a neighborhood $V \subseteq X$ of the point x such that from the conditions $G \in \gamma_i$, $G \cap V \neq \emptyset$ it follows that $G \subset U$.

[**] A finite family γ, consisting of open subsets of the space X, is called a **bordering** (^{4}) if $X \setminus |\gamma|$ is compact.

Note: Figure translations are in progress. See original paper for figures.