The asymptotic representation of the solution of a boundary value problem for a system of ordinary differential equations with a complex parameter

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Abstract

In connection with the study of one-dimensional mixed problems for second-order parabolic systems containing time derivatives in the boundary conditions, this paper provides an asymptotic representation of the solution to the spectral problem (1)-(2) for a system of ordinary differential equations outside a certain δ-neighborhood of the spectrum. Bibliography: 5 items.

Full Text

Introduction

This work, following the foundational research of M. L. Rasulov [1] and G. I. Pirmamedov [2], investigates the solution of boundary value problems for linear differential equations. We consider the problem defined by the following equation:

$$c_2(x)\frac{\partial^2 y}{\partial t^2} + c_1(x)\frac{\partial y}{\partial t} + c_0(x)y = \sum_{k=0}^{2} a_k \frac{\partial^k y}{\partial x^k} \tag{1}$$

subject to the boundary conditions:

$$a_0 y + a_1\frac{\partial y}{\partial x} + a_2\frac{\partial^2 y}{\partial x^2} = \psi(x) \tag{2}$$

where $c_k(x)$ $(k = 0, 1, 2)$ are given coefficients and $\Phi(x)$ represents the initial distribution. Following the methodology established by Rasulov [1], the problem (1)-(2) can be analyzed by examining the spectral properties of the associated differential operator.

Spectral Analysis and Asymptotic Representations

As noted by M. Z. Itskovich [3] and G. I. Pirmamedov [2], the behavior of the solution depends significantly on the roots of the characteristic equation:

$$\det(\lambda^2 E - C_2(x)) = 0 \tag{6}$$

We assume that the roots $\lambda_k(x)$ of equation (6) are distinct and satisfy specific regularity conditions for all $x \in [a, b]$. Specifically, we require that the real parts of the eigenvalues satisfy a strict ordering:

$$\mathrm{Re}\ \lambda\phi_1(x) < \mathrm{Re}\ \lambda\phi_2(x) < \dots < \mathrm{Re}\ \lambda\phi_r(x) < 0 < \mathrm{Re}\ \lambda\phi_{r+1}(x) < \dots < \mathrm{Re}\ \lambda\phi_{2r}(x) \tag{7}$$

Under these conditions, the fundamental system of solutions $y_k(x, \lambda)$ for the homogeneous equation associated with (1) admits the following asymptotic representation for large values of the spectral parameter λ:

$$\frac{d^s y_k(x, \lambda)}{dx^s} = \lambda^s [\eta_{k0s}(x) + E(x, \lambda)] \exp\left(\lambda \int_a^x \phi_k(t)dt\right) \tag{9}$$

where $s = 0, 1, 2$ and $k = 1, 2, \dots, 2r$. Here, $E(x, \lambda)$ denotes a term that vanishes as $|\lambda| \to \infty$.

Green’ s Function and Solution Representation

The solution to the boundary value problem (1)-(2) can be expressed using the Green’ s function $U(x, \xi, \lambda)$. The construction of this function involves the determinant of the boundary condition matrix, $\Delta(\lambda) = \det D(\lambda)$. For large $|\lambda|$, the determinant $\Delta(\lambda)$ can be expanded as:

$$\Delta(\lambda) = \lambda^k \exp\left(\lambda \sum_{s=1}^{\nu_s} \omega_{ks}\right) [M_0(\lambda) + E(\lambda)] \tag{19}$$

where $M_0(\lambda)$ is a leading-order term and $E(\lambda)$ represents higher-order corrections. The asymptotic behavior of the solution in various regions of the complex λ-plane is determined by the signs of Re $\lambda\omega_k$.

By applying the residue theorem and integrating over the appropriate contours in the λ-plane, we obtain the final representation of the solution $y(x, t)$. The integral representation accounts for the initial data $\Phi(x)$ and the boundary constraints. The resulting formulas allow for the effective calculation of the system’ s dynamics, extending the classical results found in [1] and [5] to a broader class of second-order differential operators with variable coefficients.

References

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3. Itskovich, M. Z., *Mathematical Modeling and Differential Equations*, 1967.
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5. Tikhonov, A. N., and Samarskii, A. A., *Equations of Mathematical Physics*, Moscow, 1964.

Note: Figure translations are in progress. See original paper for figures.