# ON INVERSION FORMULAS FOR THE FUNDAMENTAL INTEGRAL REPRESENTATION OF $p$-ANALYTIC FUNCTIONS OF $z=x+iy$ WITH CHARACTERISTIC $p=x^k$

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## Abstract

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*MATHEMATICS*

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# ON INVERSION FORMULAS FOR THE FUNDAMENTAL INTEGRAL REPRESENTATION OF $p$-ANALYTIC FUNCTIONS OF $z = x + iy$ WITH CHARACTERISTIC $p = x^k$

*(Presented by Academician I. N. Vekua, 12 I 1967)*

Let $G$ be a domain in the right half-plane $z = x + iy$, and let $a$ be a point on its boundary $S$. The fundamental integral representation of $x^k$-analytic functions of $z = x + iy$ ($x > 0,\ k = \text{const} > 0$) has the form ([1–3])

$$\begin{aligned}\tilde{f}(z) &= \tilde{u}(x,y) + i\tilde{v}(x,y)\\ &= \operatorname{Re}\int_a^z f(\zeta)\left(\frac{z+\bar{z}}{2}\right)^{1-k}(z-\zeta)^{k/2-1}(\bar{z}+\zeta)^{k/2-1}\,d\zeta\\ &\quad + i\,\operatorname{Im}\int_a^z f(\zeta)\left(\zeta - \frac{z-\bar{z}}{2}\right)(z-\zeta)^{k/2-1}(\bar{z}+\zeta)^{k/2-1}\,d\zeta,\end{aligned} \tag{1}$$

where the integration from $a$ to $z$ is carried out along any contour $\Gamma$ lying in $G$; $\arg(z-\zeta)(\bar{z}+\zeta)$ is chosen in one way or another; $f(z) = u(x,y) + iv(x,y)$ and $\tilde{f}(z) = \tilde{u}(x,y) + i\tilde{v}(x,y)$ are analytic and, respectively, $x^k$-analytic functions in the domain $G$. In this case three cases are essentially distinguished: a) $S$ contains a segment $L$ of the imaginary axis and $v|_L = 0$, while $a$ is an arbitrary (unfixed) point on $L$; b) $a = \infty$ and, as $z \to \infty$, $f(z) = O(|z|^{-k-\varepsilon})$ ($\varepsilon = \text{const} > 0$); c) $a \neq \infty$ and, as $z \to a$, $f(z) = O(|z-a|^{-1+\varepsilon})$.

Introducing the "adjoint" $p$-analytic function with characteristic $p = x^k$,

$$f^*(z) = u^*(x,y) + iv^*(x,y) = \tilde{u}(x,y) + \frac{1}{x^k}\,\tilde{v}(x,y),$$

equality (1) can be written in the equivalent form

$$f^*(z) = u^*(x,y) + iv^*(x,y)$$
$$= \frac{1}{2x^k}\int_a^z f(\zeta)(z-\zeta)^{k/2-1}(\bar{z}+\zeta)^{k/2}\,d\zeta + \overline{f(\zeta)}(\bar{z}-\bar{\zeta})^{k/2}(z+\bar{\zeta})^{k/2-1}\,d\bar{\zeta}. \tag{2}$$

From the conditions of $p$-analyticity it follows that ([4,5])

$$\frac{\partial}{\partial \bar{z}} f^* = -\frac{k}{2(z+\bar{z})}\,(f^* - \bar{f}^*), \tag{3}$$

i.e. $f^*(z)$ is a generalized analytic function ([6]) or a generalized analytic function of the fourth class ([5]).

In cases b) and c) we introduce some additional assumptions. Instead of case b) we shall speak of case b′), assuming that in $G$ there exists a rectilinear horizontal segment $C$ issuing from the point $a = \infty$, and that on this segment, as $z \to \infty$, $f(z)$, $df(z)/dz = O(x^{-2-\varepsilon})$. From case c) we single out two cases c′) and c″), assuming that in $G$ there exists a rectilinear horizontal segment $C$ issuing from the point $a$ to the right or, respectively, to the left, and that $f(z)$, $df(z)/dz = O(1)$ as $z \to a$.

It is easy to establish that

$$\widetilde{F}(z) = \widetilde{U}(x,y) + i\widetilde{V}(x,y) =$$

$$= x^{k-1}\tilde{u}(x,y) + i\int_a^z x^{1-k}\frac{\partial \tilde{v}(x,y)}{\partial x}\,dx + \left(x^{1-k}\frac{\partial \tilde{v}(x,y)}{\partial y} + (k-1)\tilde{u}(x,y)\right)dy \tag{4}$$

will be an $x^{k'}$-analytic function in $G$, where $k' = 2-k$.

If $0 < k < 2$ and

$$F^*(z) = U^*(x,y) + iV^*(x,y) = \widehat{U}(x,y) + i\frac{1}{x^k}V(x,y)$$

is the adjoint $x^{k'}$-analytic function, then the integral

$$I(z) = \int_a^z F^*(t)(t-z)^{k'/2-1}(\bar{t}+z)^{k'/2}\,dt + \overline{F^*(t)}(t-z)^{k'/2}(\bar{t}+z)^{k'/2-1}\,d\bar{t} \tag{5}$$

does not depend on the contour of integration $\Gamma$ connecting the points $a$ and $z$, and is a function of $z$ analytic in $G$.

The first part of the assertion follows from the identity (5)

$$\frac{\partial}{dt}\left(F^*(t)(t-z)^{k'/2-1}(\bar{t}+z)^{k'/2}\right) = \frac{\partial}{d\bar{t}}\left(\overline{F^*(t)}(t-z)^{k'/2}(\bar{t}+z)^{k'/2-1}\right),$$

the second part of the assertion—from the equality

$$I(z)\left(1-e^{\mp i2\pi(k'/2-1)}\right) = \int_\gamma F^*(t)(t-z)^{k'/2-1}(\bar{t}+z)^{k'/2}\,dt+$$

$$+\overline{F^*(t)}(t-z)^{k'/2}(\bar{t}+z)^{k'/2-1}\,d\bar{t},$$

where $\gamma$ is the contour going from the point $a$ along the left (or right) edge of the cut along $\Gamma$, encircling the point $z$ in the negative (or positive) direction and returning to the point $a$ along the right (or left) edge of the cut.

A. Consider cases a) and b$'$). Taking $\arg(z-\zeta)(\bar{z}+\zeta)=0$ on the upper (left) edge of the cut along $C$ and understanding by the values of the functions their values on the indicated edge of the cut along $C$ or $\Gamma$, introduce the function analytic in $G$

$$f_1(z) = u_1(x,y)+iv_1(x,y) = \mu\,\frac{1}{2e^{i\pi(k'/2-1)}}\frac{d}{dz}I(z), \tag{6}$$

where $\mu = 2\Gamma^{-1}(-k/2+1)\Gamma^{-1}(k/2)$.

On the segment $C$, equalities (6) and (1) take the form

$$f_1(z) = u_1(x,y)+iv_1(x,y) =$$

$$= \mu\frac{d}{dx}\int_{x_0}^{x}\left[\tilde{u}(\xi,y)\xi^k + i\frac{x}{\xi^{2-k}}V(\xi,y)\right]\frac{d\xi}{(x^2-\xi^2)^{k/2}}, \tag{7}$$

$$\tilde{f}(z) = \tilde{u}(x,y)+i\tilde{v}(x,y) = \int_{x_0}^{x}\left[x^{1-k}u(\xi,y)+i\xi v(\xi,y)\right](x^2-\xi^2)^{k/2-1}\,d\xi, \tag{8}$$

where $x_0 = \operatorname{Re} a$.

From (8) and (4) it follows that

$$\tilde{v}(x,y)\big|_{z=a} = 0, \qquad \widetilde{V}(x,y)\Big|_{z=a} = 0. \tag{9}$$

The inversion formula (8) has the form ([5,7])

$$u(x,y)+ixv(x,y)=\mu\frac{d}{dx}\int_{x_0}^{x}\left[i\tilde{u}(\xi,y)\xi^{k-1}+\tilde{v}(\xi,y)\right]\frac{\xi\,d\xi}{(x^2-\xi^2)^{k/2}}. \qquad (10)$$

Comparing (7), (10) and taking (9) into account, we find on $C$

$$f_1(z)=f(z)=\mu\frac{d}{dx}\int_{x_0}^{x}\tilde{u}(\xi,y)\,\frac{\xi^k\,d\xi}{(x^2-\xi^2)^{k/2}}+i\mu x^{1-k}\int_{x_0/x}^{1}\frac{\partial\tilde{v}(x\beta,y)}{\partial x\beta}\,\frac{d\beta}{(1-\beta^2)^{k/2}}.$$

Consequently, $f_1(z)\equiv f(z)$ in the domain $G$, and the desired inversion formula for the basic integral representation (1) is written in the form

$$f(z)=u(x,y)+iv(x,y)= \qquad (11)$$

$$=\mu\frac{1}{2e^{i\pi(k'/2-1)}}\frac{d}{dz}\int_a^z F^*(t)(t-z)^{k'/2-1}(\bar{t}+z)^{k'/2}\,dt+\bar{F}^*(t)(t-z)^{k'/2}(\bar{t}+z)^{k'/2-1}\,d\bar{t}$$

or

$$f(z)=u(x,y)+iv(x,y)=\mu\frac{1}{2e^{i\pi(k'/2-1)}}\frac{d}{dz}\int_a^z\widetilde{U}(\xi,\eta)\,d\widetilde{Z}+i\widetilde{V}(\xi,\eta)\,dZ, \qquad (12)$$

where $Z$ and $\widetilde{Z}$ are the conjugate variables ([4,5]) of $t=\xi+i\eta$ with characteristic $p=\xi^{k'}$,

$$\widetilde{Z}=\frac{2}{k'}(t-z)^{k'/2}(\bar{t}+z)^{k'/2},\quad dZ=2^{k'+1}\left(\sqrt{\frac{t-z}{t+z}}+\sqrt{\left(\frac{\bar{t}+z}{t-z}\right)^{-k'}}\right)d\ln\sqrt{\frac{t-z}{\bar{t}+z}}. \qquad (12')$$

B. Let us consider cases $'$) and $''$). Setting $\arg(z-\xi)(\bar{z}+\xi)=-\pi$ on the upper (right-hand) edge of the cut along $C$ and understanding by the values of the functions their values on the indicated edge of the cut along $C$ or $\Gamma$, introduce the function analytic in $G$

$$f_2(z)=u_2(x,y)+iv_2(x,y)=\mu\frac{1}{2}\frac{d}{dz}I(z). \qquad (13)$$

From (13) and (1) we obtain on $C$

$$f_2(z) = u_2(x,y) + iv_2(x,y) = \mu \frac{d}{dx} \int_{x_0}^{x} \left[ \tilde{u}(\xi,y)\xi^k + i\frac{x}{\xi^{2-k}}\widetilde{V}(\xi,y) \right] \frac{d\xi}{(\xi^2 - x^2)^{k/2}}, \quad (14)$$

$$\tilde{f}(z) = \tilde{u}(x,y) + i\tilde{v}(x,y) = \int_{x_0/x}^{1} [u_3(x\beta,y) + ix^k\beta v_3(x\beta,y)](\beta^2-1)^{k/2-1}\,d\beta, \quad (15)$$

where $f_3(z) = u_3(x,y) + iv_3(x,y) = e^{-i\pi(k/2-1)}f(z)$. From (15) it follows that in case $'$), when approaching infinity,

$$\tilde{u}(x,y), \quad \partial\tilde{u}(x,y)/\partial x, \quad \partial\tilde{u}(x,y)/\partial y = O(x^{-2-\varepsilon}), \quad \tilde{v}(x,y) = O(x^{k-2-\varepsilon}).$$

Therefore in case $'$), as in case $''$), the equalities (9) hold. Taking this and the inversion formula for the basic integral representation (1) on $C$ ([5],[7] ) into account,

$$u_3(x,y) + ixv_3(x,y) = \mu \frac{d}{dx} \int_{x_0}^{x} [\tilde{u}(\xi,y)\xi^{k-1} + i\tilde{v}(\xi,y)] \frac{\xi\,d\xi}{(\xi^2 - x^2)^{k/2}},$$

we come to the conclusion that on $C$

$$f_2(z) = f_3(z) = \mu \frac{d}{dx} \int_{x_0}^{x} \tilde{u}(\xi,y)\,\frac{\xi^k\,d\xi}{(\xi^2 - x^2)^{k/2}} + i\mu x^{1-k} \int_{x_0/x}^{1} \frac{\partial\tilde{v}(x\beta,y)}{\partial x\beta} \frac{d\beta}{(\beta^2 - 1)^{k/2}}.$$

Thus, the desired inversion formula for the basic integral representation (1) is obtained in the following form:

$$f(z) = u(x,y) + iv(x,y) =$$

$$= \mu \frac{1}{2e^{i\pi k'/2}} \frac{d}{dz} \int_a^z F^*(t)(t-z)^{k'/2-1}(\bar{t}+z)^{k'/2}\,dt + \bar{F}^*(t)(t-z)^{k'/2}(\bar{t}+z)^{k'/2-1}\,d\bar{t} \quad (16)$$

or

$$f(z) = u(x,y) + iv(x,y) = \mu \frac{1}{2e^{i\pi k'/2}} \frac{d}{dz} \int_a^z \widetilde{U}(\xi,\eta)\,d\widetilde{Z} + i\widetilde{V}(\xi,\eta)\,dZ, \quad (17)$$

where $Z$ and $\widetilde{Z}$ are conjugate variables defined by the equalities (12′), under the agreement made above on the choice of $\arg(z-\zeta)(\bar{z}+\zeta)$.

C). Let now $k = 2m + k_1$, where $m$ is a positive integer, $0 < k_1 < 2$. From (2) it follows that the function

$$f_1^*(z) = u_1^*(x,y) + iv_1^*(x,y) = \frac{1}{\alpha x^{k_1}} \frac{d^{2m}}{dz^m\, d\bar{z}^m}(x^k f^*(z)),$$
$$\alpha = \left(\frac{k}{2}-1\right)\frac{k}{2}\left(\frac{k}{2}-2\right)\left(\frac{k}{2}-1\right)\cdots\frac{1}{2}\,\frac{3}{2} \tag{18}$$

will be adjoint $x^{k_1}$-analytic in $G$. Introducing the $x^{k_1}$-analytic function

$$f_1(z) = \tilde{u}_1(x,y) + i\tilde{v}_1(x,y) = u_1^*(x,y) + ix^{k_1}v_1^*(x,y),$$

we arrive at the conclusion that the inversion formula of the principal integral representation (1) for $k = 2m + k_1$ is given by equality (11) or (12) in cases a) and b′), and by equality (16) or (17) in cases b) and c′), provided that in these equalities $k$ is replaced by $k_1$, and $\tilde{u}(x,y)$ and $\tilde{v}(x,y)$ are replaced by $\tilde{u}_1(x,y)$ and $\tilde{v}_1(x,y)$.

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*Note: Figure translations are in progress. See original paper for figures.*