ON A CLASS OF STRUCTURES WITH OPERATORS AND A NEW CHARACTERISTIC OF THE POSITIVE PART OF A \(K\)-SPACE

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M. G. RABINOVICH

ON A CLASS OF STRUCTURES WITH OPERATORS AND A NEW CHARACTERISTIC OF THE POSITIVE PART OF A K-SPACE

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In the work of A. G. Pinsker ([1]) a characteristic of the cone of positive elements of a K-space is given, and it is proved that, in essence, this object is determined only by the properties of the additive operation and of the order relation ([2]). In the present note it is shown that in the characteristic of the positive part of a K-space one may dispense altogether with the definition of the additive operation, relying on the properties of the order and of the operation of multiplication by a number. At the same time, some properties of one class of structures with operators are investigated. Where no special definitions are given, all terminology is borrowed by us from the book ([3]).

Definition 1. A set X is called an (l, λ)-system if the following conditions are satisfied:

L1. X is a conditionally complete lattice with least element 0.

L2. On X there is defined an external law of composition $(\lambda, x) \to \lambda x$, $R^1_+ \times X \to X$, where R^1_+ is the set of nonnegative real numbers, with

$$\lambda(\mu x) = (\lambda\mu)x, \quad 1 \cdot x = x.$$

L3. From $x \geq y,\ \lambda \geq \mu$ it follows that $\lambda x \geq \mu y$, and, if $nx \leq y$ for every n, then $x = 0$.

Example 1. The cone of positive elements of a K-space.

Example 2. The collection, ordered by inclusion, of all convex linearly bounded ([4]) subsets of a vector space that contain the zero vector.

Example 3. The collection, ordered by inclusion, of all closed linearly bounded star-shaped subsets of a topological vector space that contain the zero vector. A set A is called star-shaped if from $a \in A$ it follows that $\lambda a \in A$ for all $0 \leq \lambda \leq 1$.

Example 4. The set of real functions on $[0,1]$ generated by the two functions $f_1(x) = 1$ and $f_2(x) = x$ by means of the operations of multiplication by a number, sup, and inf.

Lemma 1. a) If $x = \sup_\alpha x_\alpha$, then $\lambda x = \sup_\alpha \lambda x_\alpha$; b) if $y = \inf_\beta y_\beta$, then $\mu y = \inf_\beta \mu y_\beta$; c) if $x\, d\, y$, i.e. $x \wedge y = 0$, then $\lambda x\, d\, \mu y$ for arbitrary $\lambda, \mu \in R^1_+$.

Lemma 2. a) If $x \neq 0$, then $\lambda x > \mu x$ for arbitrary $\lambda > \mu$; b) if $x > y$, then $\lambda x > \lambda y$ for arbitrary $\lambda > 0$.

Definition 2. A subset X_1 of an (l, λ)-system X is called a component in X if X_1 is a proper and normal sublattice in X, and the operation of multiplication by a number does not lead outside X_1.

In order that the concept of a component in an (l, λ)-system be meaningful, we introduce the following axiom:

D. If $x \leq \sup_\alpha y_\alpha$ and $x\, d\, y_{\alpha_0}$, then

$$x \leq \sup_{\alpha \neq \alpha_0} y_\alpha.$$

Let us note that this condition is weaker than the infinite distributive law

$$x \wedge \sup_\alpha y_\alpha = \sup_\alpha (x \wedge y_\alpha).$$

Axiom D is satisfied in all the examples except Example 2.

Lemma 3. a) The disjoint complement of any set in X is a component in X; b) in X there exists a complete set of pairwise disjoint components.

Definition 3. Let X_0 be a component in X. For any $x \in X$ put

$$\mathrm{pr}_{X_0} x = \sup\{y :\ y \in X_0,\ y \leq x\}.$$

Lemma 4. Let X_0 be a component in X. Then: a) $\mathrm{pr}_{X_0} x \leq x$, $\mathrm{pr}_{X_0} \mathrm{pr}_{X_0} x = \mathrm{pr}_{X_0} x$, $x \in X_0$ is equivalent to $\mathrm{pr}_{X_0} x = x$; b) $x\, d\, y$ implies $\mathrm{pr}_{X_0} x\, d\, \mathrm{pr}_{X_0} y$; $x\, d\, X_0$ is equivalent to $\mathrm{pr}_{X_0} x = 0$.

As examples show, the single condition D does not ensure "good" properties of the projection operator in an (l, λ)-system. Therefore we introduce two more axioms.

A1. If $z \vee y > y$, then there is a component $X_\alpha \subset X$ such that $\mathrm{pr}_\alpha y \leq z$.

A2. If $x > y$, then there is z: $z \vee y > y$ and $x > \lambda z$, where $\lambda > 1$.

Theorem 1. Let the (l, λ)-system X satisfy D, A1, A2, and let the system of components $\{X_\alpha\}$ form a decomposition of X ([3]). Then for every $x \in X$ the equality $x = \sup_\alpha \{\mathrm{pr}_\alpha x\}$ holds. If, however, $x = \sup_\alpha x_\alpha$, where $x_\alpha \in X_\alpha$, then $x_\alpha = \mathrm{pr}_\alpha x$.

Definition 4. An element $a \in X$ is called a **unit** if $X_a = X$, where X_a is the component generated by a ([3]). The projection of a unit onto a component is called a **unit element**.

Lemma 5. In X there exists a complete set of pairwise disjoint components with units.

Let now X contain a unit.

Lemma 6. For every $x \in X$ there exist such a unit element e and a number $\alpha \geq 0$ that $x > \alpha e$.

Let x be an arbitrary element of X. Denote

$$K_n(x) = \left\{ \bigvee_{i=1}^{n} \lambda_i e_i : \; e_i \, d \, e_j, \; \bigvee_{i=1}^{n} \lambda_i e_i \leq x \right\}.$$

Here $\{e_i\}_{1 \leq i \leq n}$ are all possible decompositions of the unit into n unit elements, and for a given e_i

$$\lambda_i = \sup\{\lambda : \; \lambda e_i \leq x\}.$$

We note that it is possible that $\lambda_i = 0$. Denote

$$K(x) = \bigcup_{n=1}^{\infty} K_n(x).$$

Theorem 2. For every $x \in X$

$$x = \sup K(x).$$

We now define an addition operation in X. Let $x, y \in X$. Put

$$K_n(x, y) = \left\{ \bigvee_{i=1}^{n} (\lambda_i + \mu_i) e_i : \; \bigvee_{i=1}^{n} \lambda_i e_i \in K_n(x), \; \bigvee_{i=1}^{n} \mu_i e_i \in K_n(y) \right\};$$

$$K(x, y) = \bigcup_{n=1}^{\infty} K_n(x, y).$$

Then, by definition,

$$x + y = \sup K(x, y).$$

Lemma 7. The addition operation in X satisfies the following conditions:

$$x + y = y + x, \qquad (x + y) + z = x + (y + z),$$

$$(\lambda + \mu)x = \lambda x + \mu x, \qquad \lambda(x + y) = \lambda x + \lambda y, \qquad 0 + x = x.$$

Lemma 8. a) $x > y$ is equivalent to $x = y + z,\ z \neq 0$; b) $x + y \leq y + u$ implies $x \leq u$.

Theorem 3. Let X be an (l, λ)-system satisfying D, A1, A2. Then X is, up to isomorphism, the cone of positive elements of a K-space.

The proof of this theorem essentially relies on the theorem of the work ([1]) on a characterization of the positive part of a K-space.

Remark. The conditions D and A1, A2 are independent of one another, as follows from the examples given below.

Example 5. The inclusion-ordered collection of convex closed bounded subsets of the space R^n that contain zero is an (l, λ)-system with conditions A1, A2, but without condition D.

Example 6. Let X be the join (3) of two components X_1 and X_2, where X_1 is the function space of Example 4, and X_2 is the set of nonnegative measurable functions on $[1, 2]$. Then X is an (l, λ)-system with D, but without A1 and A2.

Let us note that a trivial example of this kind is the (l, λ)-system of Example 4 itself.

Example 7. The ordered collection of all bounded star-shaped subsets of R^n that contain zero is an (l, λ)-system with D and A1, but without A2. Here the exact upper bound of a bounded collection of star-shaped subsets is taken to be their union (which is a bounded star-shaped subset), and the exact lower bound is their intersection.

Since, obviously, conditions D, A1, A2 are necessary in a K-space, Theorem 3 gives a characterization of the cone of positive elements of a K-space in terms of the order relation and multiplication by a number.

Leningrad Engineering-Economic Institute
named after Palmiro Togliatti

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CITED LITERATURE

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3. L. V. Kantorovich, B. Z. Vulikh, A. G. Pinsker, *Functional Analysis in Semi-Ordered Spaces*, Moscow—Leningrad, 1950.

4. V. Klee, *Proc. Am. Math. Soc.*, **7**, 4 (1956).

Note: Figure translations are in progress. See original paper for figures.