THE CAUCHY PROBLEM AND A BOUNDARY-VALUE PROBLEM FOR LINEAR AND QUASILINEAR DEGENERATE HYPERBOLIC EQUATIONS OF SECOND ORDER

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THE CAUCHY PROBLEM AND A BOUNDARY-VALUE PROBLEM FOR LINEAR AND QUASILINEAR DEGENERATE HYPERBOLIC EQUATIONS OF SECOND ORDER

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The Cauchy problem and the boundary-value problem for linear hyperbolic equations of second order degenerating in the domain and on the boundary were studied in the work of O. A. Oleinik (^{1}). In the case of an equation degenerating only on the boundary of the domain, these problems have been considered by many authors (see the literature cited in (^{1})).

I. The Cauchy problem. In the domain $G_T = \{0 \le t \le T,\ x \in R_n(x_1, \dots, x_n)\}$ we consider the Cauchy problem for the equation

$$Lu \equiv -u_{tt} + a^{ij}(t,x)u_{x_i x_j} + b^i(t,x)u_{x_i} + b^0(t,x)u_t + c(t,x)u = f(t,x), \quad (1)$$

where $a^{ij} = a^{ji}$, $a^{ij}(t,x)\xi_i\xi_j \ge 0$ *, for $(t,x) \in G_T$ and all $\xi = (\xi_1, \dots, \xi_n)$, with zero initial conditions

$$u(0,x) = 0, \qquad u_t(0,x) = 0. \quad (2)$$

For simplicity it is assumed that $f(t,x)$ is a finite function with respect to x.

Applying the methods of the works (^{1},^{2}), as well as the methods developed by S. L. Sobolev (^{3}), we shall indicate smoothness conditions on the coefficients of equation (1) which ensure the existence of a classical solution of the problem (1), (2), which also permits consideration of quasilinear equations of the form

$$-u_{tt} + a^{ij}(t,x,u)u_{x_i x_j} + b^i(t,x,u)u_{x_i} + b^0(t,x,u)u_t = f(t,x,u), \quad (3)$$

$$a^{ij}(t,x,u)\xi_i\xi_j \ge 0 \quad \text{for } (t,x) \in G_T, \quad |u| \le M, \quad M = \text{const} > 0.$$

Let

$$D = \partial^k/\partial t^{k_0}\partial x_1^{k_1}\dots\partial x_n^{k_n} \quad (k_0+\dots+k_n = k), \qquad D_x^k = \partial^k/\partial x_1^{k_1}\dots\partial x_n^{k_n} \quad (k_1+\dots+k_n = k).$$

Theorem 1. In order that the Cauchy problem for equation (1) with conditions (2) have a solution continuous together with derivatives up to order $m \geq 2$, it is sufficient that the following conditions be satisfied for $l \geq m + 1 + [(n+1)/2]$:

1) The coefficients a^{ij} and their derivatives $a^{ij}_{x_k}$, a^{ij}_t, $a^{ij}_{x_k x_r}$ are bounded:

$$\max\{|a^{ij}|,\ |a^{ij}_t|,\ |a^{ij}_{x_k}|,\ |a^{ij}_{x_k x_r}|\} \leq A,$$

the derivatives of a^{ij} of orders $2, 3, \dots, l$ satisfy the inequalities

$$\max\{\|D^\beta a^{ij}\|_{L_{p_1}(G)},\ \|D_x^\beta a^{ij}\|_{L_{p_2}(G)}\} \leq A_{G_T} \leq A,$$

where $p_s = (n+1+\sigma)/(\beta-s)$ for $\beta \leq s+(n+1)/2$, $p_s = 2$ for $\beta > s+(n+1)/2$; $s = 1, 2;\ \sigma > 0$.

* Here, as everywhere below, summation over repeated indices from 1 to n is assumed.

2) The coefficients $b^i(t,x)$ $(i = 0, \dots, n)$ and their derivatives $b^i_{x_k}$, b^0_t are bounded:

$$\max\{|b^i|,\ |b^0_t|,\ |b^i_{x_k}|\} \leq A;$$

moreover, the derivatives of $b^i(t,x)$ up to order l satisfy the inequalities

$$\left\{\|D^\beta b^i\|_{L_{q_0}(G_T)},\ \|D_x^\beta b^i\|_{L_{q_1}(G_T)}\right\} \leq A_{G_T} \leq A,$$

where

$$q_s = (n+1+\sigma)/(\beta-s) \quad \text{for } \beta \leq s+(n+1)/2, \qquad q_s = 2 \quad \text{for } \beta > s+(n+1)/2;$$

$s = 0, 1;\ \sigma > 0.$

3) The coefficient $c(t,x)$ is bounded:

$$\max|c(t,x)| \leq A,$$

and the derivatives of $c(t,x)$ up to order l satisfy the inequalities

$$\left\{\|D^\beta c\|_{L_{r_1}(G_T)},\ \|D^\beta c\|_{L_{r_0}(G_T)}\right\} \leq A_{G_T} \leq A,$$

where

$$r_s = (n+1+\sigma)/(\beta+s) \quad \text{for } \beta \leq (n+1)/2-s, \qquad r_s = 2 \quad \text{for } \beta > (n+1)/2-s;$$

$s = 0, 1;\ \sigma > 0.$

4) The function $f(t,x)$ and its derivatives up to order l satisfy the inequalities

$$\|D^\beta f(t,x)\|_{L_2(G_T)} \le F_{G_T} \le F. \qquad D^i f(0,x) = 0, \quad 0 \le i \le l-2.$$

In addition, it is assumed that there exist $K > 0$ and $\delta > 0$ such that

$$K a^{ij}\xi_i\xi_j + a^{ij}_t \xi_i\xi_j - \delta\big[(b^i - a^{ij}_{x_j})\xi_i\big]^2 \ge 0 \quad \text{for all } \xi. \tag{4}$$

Remark. Inequality (4) is essential for the correctness of the problem (1), (2) (see (^{4},^{5})).

The proof of Theorem 1 is carried out according to the same scheme as in (^{1}). The main point of the proof is obtaining inequalities of the form

$$U_k(t) \le C \int_0^t U_k^{1/2}(\tau) \left\{ A \sum_{i=0}^{k} U_i^{1/2}(\tau) + F \right\} d\tau, \tag{5}$$

where

$$U_k(\tau) = \iint_{G_\tau} \sum_{k_0+\dots+k_n=k} \left(\frac{\partial^k u}{\partial t^{k_0} \cdots \partial x_n^{k_n}} \right)^2 d\tau\, dx, \qquad C = \text{const}.$$

In proving these inequalities one repeatedly uses the embedding theorems of S. L. Sobolev, similarly to (^{3}) (see § 21).

Consider the Cauchy problem for the quasilinear equation (3) with conditions (2). The function $f(t,x,u)$ is assumed to be finite with respect to x, and the support of $f(t,x,u)$ for $t \le T$, $|u| \le M$ belongs to the cylinder

$$Q^0 = \{[0,T] \times \Omega_0\}.$$

Let Ω_1 be a bounded domain such that $\Omega_1 \supset \Omega_0$, and let the distance between the boundaries of Ω_1 and Ω_0 be not less than

$$\rho_0 = 1 + T\sqrt{n(A_0+1)},$$

where

$$A_0 = \sup |a^{ij}(t,x,u)|$$

for $(t,x) \in G_T$, $|u| \le M$. Let

$$Q_\tau = \{0 \le t \le \tau,\ x \in \Omega_1\}, \qquad 0 < \tau \le T.$$

Let the function $\Phi(t, x_1, \dots, x_n, u)$ be defined in the $(n+2)$-dimensional domain

$$E \equiv \{0 \le t \le T,\ x \in \Omega_1,\ |u| \le M\},$$

be continuous in E, and have l continuous derivatives with respect to u, while the functions

$$\Phi_{u^k} \equiv \frac{\partial^k}{\partial u^k}\Phi \qquad (0 \le k \le l)$$

have generalized derivatives up to order l with respect to $t, x_1, \dots, x_n$ for each fixed value of u. Denote by E_u the set of functions continuous in $\overline{Q_\tau}$:

$$u = v(t,x), \qquad |v(t,x)| \le M.$$

Definition. We shall say that the function $\Phi(t,x,u)$ has property $\widetilde{T}$ if there exists a number $p > 1$, $p > (n+1)/l$, such that the result of substituting into the function

$$D^\alpha \Phi_{u^k} \equiv \frac{\partial^\alpha}{\partial t^{\alpha_0} \cdots \partial x_n^{\alpha_n}} \Phi_{u^k}(t,x,u)$$

instead of u any function from E_u is a composite function of (t,x) belonging, for

$$l \ge \alpha > l - (n+1)/p,$$

to the space

$$L_{\frac{1}{1/p-(l-\alpha)/(n+1)}}(Q_\tau),$$

and moreover

$$\left\| D^\alpha \Phi_{u^k}(t,x,u)\big|_{u=v(t,x)} \right\|_{L_{\frac{1}{1/p-(l-\alpha)/(n+1)}}} \le A_{Q_\tau},$$

where the constant A_{Q_τ} does not $\cdots$

depends on $v(t,x) \in E_u$, $0 \le \tau \le T$. If $\alpha < l < (n+1)/p$, then the space

$$L_{\frac{1}{1/p-(l-\alpha)/(n+1)}}$$

should be replaced by $C(Q_\tau)$, and for $\alpha = l - (n+1)/p$, respectively, by L_q, where $q > 1$ is arbitrary (the constant A_{Q_τ} depends on q).

Lemma 1. If the function $\Phi(t,x,u)$ has property $\widetilde{T}$, and the function $u = v(t,x)$ satisfies the condition

$$\iiint_{Q_\tau} |D^\alpha v(t,x)|^{\frac{1}{1/p-(l-\alpha)/(n+1)}}\, dt\, dx \le B_{Q_\tau}^{\frac{1}{1/p-(l-\alpha)/(n+1)}},$$

then for the total derivatives of the function $\Phi(t,x,v(t,x))$ the estimates

$$\left\{ \iiint_{Q_\tau} \left| \frac{\partial^\alpha \Phi(t,x,v(t,x))}{\partial t^{\alpha_0} \cdots \partial x_n^{\alpha_n}} \right|^{\frac{1}{1/p-(l-\alpha)/(n+1)}} dt\, dx \right\}^{1/p-(l-\alpha)/(n+1)} \le C A_{Q_\tau}\{B_{Q_\tau}^\alpha + 1\},$$

hold, where C is a constant independent of $v(t,x)$.

This lemma is analogous to the theorem of S. L. Sobolev (see (^{3}), p. 230); its proof is based on the application of the inequalities of Hölder and Minkowski.

Theorem 2. If the coefficients and the right-hand side of equation (3), as functions of the variables (t, x, u), satisfy in Q_T condition $\widetilde{T}$ with $p = 2$, $l = [(n+1)/2] + 3$, and if there exist $K > 0$, $\delta > 0$, $M_1 > 0$, $M_2 > 0$ such that for the coefficients of equation (3) the inequality

$$Ka^{ij}\xi_i\xi_j + a_t^{ij}\xi_i\xi_j - \delta\left[(b^i - a^{ij}_{x_j})\xi_i\right]^2 - M_1(a^{ij}_u\xi_i\xi_j) - \delta M_2^2\sum_{j=1}^{n}(a^{ij}_u\xi_i)^2 \ge 0 \quad (6)$$

is fulfilled for $(t, x) \in Q_T$, $|u| \le M$, then one can specify $t_0 \le T$ such that in the cylinder Q_{t_0} there exists a solution of problem (3), (2), continuous with derivatives up to order two inclusive; moreover, $u \in W_2^l(Q_{t_0})$ and

$$|u(t,x)| \le M, \qquad |u_t(t,x)| \le M_1, \qquad |u_{x_i}(t,x)| \le M_2 \quad (7)$$

in Q_{t_0}. The number t_0 depends on M, M_1, M_2.

The proof of the theorem is carried out by the method of successive approximations. For the coefficients of the equation

$$L_N(u^N) \equiv -u^N_{tt} + a^{ij}(t,x,u^{N-1})u^N_{x_ix_j} + b^i(t,x,u^{N-1})u^N_{x_i} + b^0(t,x,u^{N-1})u^N_t = f(t,x,u^{N-1}) \quad (8)$$

by virtue of condition $\widetilde{T}$, Lemma 1, and embedding theorems we obtain

$$\max\left\{\left\|\frac{\partial^\alpha a^{ij}(t,x,u^{N-1}(t,x))}{\partial t^{\alpha_0}\cdots\partial x_n^{\alpha_n}}\right\|_{L_q(Q_\tau)}, \left\|\frac{\partial b^i(t,x,u^{N-1}(t,x))}{\partial t^{\alpha_0}\cdots\partial x_n^{\alpha_n}}\right\|_{L_q(Q_\tau)},\right.$$

$$\left.\left\|\frac{\partial^\alpha f(t,x,u^{N-1}(t,x))}{\partial t^{\alpha_0}\cdots\partial x_n^{\alpha_n}}\right\|_{L_q(Q_\tau)}\right\} \le C\{1 + [U^{(N-1)}(\tau)]^{1/2}\},$$

where

$$\frac{1}{q} = \frac{1}{2} - \frac{l-\alpha}{n+1}, \qquad l = 3 + \left[\frac{n+1}{2}\right], \qquad U^{(N-1)}(\tau) = \sum_{\alpha\le l}\iiint_{Q_\tau}\left(\frac{\partial^\alpha u^{N-1}}{\partial t^{\alpha_0}\cdots\partial x_n^{\alpha_n}}\right)^2 dt\,dx.$$

Applying the theorem on the complete continuity of the embedding operator $W_2^l(Q_\tau)$ into $C_2(Q_\tau)$ $(l = 3 + [(n+1)/2])$ (see (^{3}), p. 93) and inequalities of the form (5) to the solution $u^N(t,x)$ of problem (8), (2), one can prove that there exists a sufficiently small number $t_1(M, M_1, M_2, M_3) > 0$ such that for $\tau \le t_1$ equation (8) with conditions (2) defines a mapping of the set of functions $u^{N-1} \in W_2^l(Q_\tau)$, satisfying the initial conditions (2), the inequal-

properties (7) and such that $U^{(N-1)}(\tau) \le M_3$, where $M_3 = \text{const} > 0$, into itself. By virtue of the embedding theorems, this set is compact in $C_2^2(Q_\tau)$. Moreover, considering the equation for the difference $v^N = u^N - u^{N-1}$, one can show that for $\tau \le t_0(M, M_1, M_2, M_3, \theta)$ the inequality $\|v^N\|_{L_2(Q_\tau)} \le \theta \|v^{N-1}\|_{L_2(Q_\tau)}$ will hold, where $\theta < 1$, whence it follows that the sequence $\{u^N\}$ converges in the norm $L_2(Q_\tau)$ for $t_0 \le t_1$. The limiting function will be the desired solution of problem (3), (2) in the cylinder Q_{t_0}.

II. **Boundary-value problem.** In the cylinder $Q^T = \{[0,T] \times \Omega\}$, for equation (1) we consider the mixed problem with initial conditions (2) and the boundary condition

$$u(t,x)\big|_S = 0, \tag{9}$$

where $S = \{[0,T] \times \Sigma\}$, and Σ is the boundary of Ω.

Problem (1), (2), (9) was studied in papers (^{1},^{6}).

We shall say that the function $f(t,x)$ belongs in Q^T to the class of functions W_0 if $f(t,x)$ is the closure in the norm $W_2^l(Q^T)$ of functions $f_\varepsilon(t,x)$, infinitely differentiable in $\overline{Q^T}$, equal to zero in an ε-neighborhood of Ω, where $l = m + 1 + [(n+1)/2]$.

Theorem 3. *Suppose that in the cylinder Q^T the assumptions of Theorem 1 concerning the coefficients of equation (1) are satisfied for $l \ge m+1+[(n+1)/2]$. In addition, suppose that*

$$a^{ij}(t,x)\xi_i\xi_j \ge \mu\xi_i\xi_i \quad \text{for } (t,x) \in S$$

and all $\xi = (\xi_1, \dots, \xi_n)$, $\mu = \text{const} > 0$; $f(t,x) \in W_0$, and the boundary Σ of the domain Ω belongs to the class A^{l+2}. Then there exists a solution of problem (1), (2), (9), continuous in Q^T with derivatives up to order $m \ge 2$ inclusive, and $u(t,x) \in W_2^l(Q_T)$.

The basic point of the proof, as in Theorem 1, is the derivation of inequalities of the form (5). We note that in proving these inequalities for domains lying inside Q^T the same methods are applied as in Theorem 1, while for estimates in the boundary strip one uses methods analogous to those set out in (7), Ch. III, § 3, and embedding theorems.

We transform the quasilinear equation (3) to the form

$$-u_{tt} + a^{ij}(t,x,u)u_{x_ix_j} + b^i(t,x,u)u_{x_i} + b^0(t,x,u)u_t + c(t,x,u)u = f(t,x). \tag{10}$$

Theorem 4. *Suppose that for the coefficients of equation (10) in Q^T the assumptions of Theorem 2 are satisfied. In addition, suppose that*

$$a^{ij}(t,x,u)\xi_i\xi_j \ge \mu\xi_i\xi_i \quad \text{for } (t,x) \in S,\ |u| \le M \text{ and all } \xi = (\xi_1, \dots, \xi_n),$$

where $\mu = \text{const} > 0$*;* $f(t,x) \in W_0$*, and the boundary* $\Sigma \in A^{l+2}$*. Then one can specify a* $t_1(M, M_1, M_2)$ *such that in the cylinder* Q^{t_1} *there exists a classical solution* $u(t,x)$ *of problem (10), (2), (9); moreover,* $u \in W_2^l(Q^{t_1})$*, and the inequalities (7) hold for it.*

The proof is carried out by the method of successive approximations, as in Theorem 2, using the a priori inequalities proved in Theorem 3 for the linear equation and embedding theorems.

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Note: Figure translations are in progress. See original paper for figures.

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