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Abstract

Full Text

MATHEMATICS

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RINGS OF OPERATIONS AND ADAMS-TYPE SPECTRAL SEQUENCES IN EXTRAORDINARY COHOMOLOGY THEORIES.

U-COBORDISMS AND k-THEORY

I. Let $X = (X_n)$ be a spectrum of $(n-1)$-connected complexes and maps $f_n : EX_n \to X_{n+1}$ which are homotopy equivalences up to large dimension. $X^*(K, L)$ denotes the cohomology of the pair (K, L) with values in X. By $H^i(X, Z)$ we denote, naturally, $\lim_{n\to\infty} H^{n+i}(X_n, Z)$, and by

$$X^*(X) = \lim_{n\to\infty} X^i(X_n), \qquad \text{where } X^*(X) = \sum_i X^i(X),$$

the Steenrod ring A^X.

One may consider the A^X-modules of cohomology

$$X^*(K) = \sum_i X^i(K).$$

By analogy with the work ([2]), one can sometimes construct a spectral sequence (E_r, d_r), where

$$E_2 = \mathrm{Ext}_{A^X}(X^*(K), X^*(L)).$$

The question arises: when does this spectral sequence converge to $\mathrm{Map}^s[L, K]$? We shall be interested in this question for $L = S^0$.

Theorem 1. If for a point P the group $X^0(P) = Z$, the cohomology $H^*(X, Z)$ has no torsion, and the spectral sequence (E_r, d_r) with term

$$E_2 = H^*(K, X^*(P)),$$

converging to $X^*(K)$, has all differentials d_i equal to zero for all complexes K whose cohomology has no torsion, then for all such complexes K the Adams spectral sequence with term

$$E_2 = \mathrm{Ext}_{A^X}(X^*(K), X^*(P))$$

exists and converges precisely to the stable homotopy groups $\pi_*^s(K)$.

Corollary 1. For the spectrum $X = MU$, $X_{2n} = MU_n$, the Adams spectral sequence for any complex without torsion exists and converges to the stable homotopy groups $\pi_*^s(K)$.

Remark. For the spectrum $X = k$, $X_0 = BU \times Z$, $X_{2n} = BU^{(2n)}$, where $BU^{(2n)}$ is the $(2n-1)$-connected space BU, the conditions of Theorem 1 are not satisfied. The spectrum k is the spectrum of "stable" K-theory; here

$$\Omega^{2n} X_{2n} = BU \times Z$$

(ordinary K-theory, where $X_{2n} = BU \times Z$ and $X_{2n-1} = U$, do not satisfy the requirement of "stabilization" $\pi_{i-k}(X_i) = 0$).

II. The main question is the following: how are the Steenrod rings A^U and A^k to be computed for the theories of U-cobordisms and k-theory, respectively? We shall first compute the ring A^U.

Consider the cobordism ring

$$\Omega_U = U^*(P) = Z[x_1, \ldots, x_i, \ldots],$$

$\dim x_i = -2i$ (see ([9–11])). The ring $X^*(K)$, $X = MU$, will always be denoted by $U^*(K)$. Since $\Omega_U = U^*(P)$, we have multiplication operations on the cohomology of a point $x: \ U^j(K, L) \to U^{j-2k}(K, L)$, $\dim x = -2k$, $x \in \Omega_U^{2k}$, where $\Omega_U^{2k} = U^{-2k}(P)$. Obviously, for any $\alpha, \beta \in U^*(K, L)$ we have

$$x(\alpha\beta) = (x\alpha)\beta = \alpha(x\beta).$$

We shall henceforth regard the ring A^U as a left Ω_U-module. This module is free.

We note further that

$$N = A^U \otimes_{\Omega_U} A^U$$

is also a left A^U-module; since $N = U^*(MU \wedge MU)$. By virtue of the multiplication $MU \wedge MU \to MU$, a "diagonal"

$$\Delta : \ A^U \to A^U \otimes_\Omega A^U$$

is defined.

In U-theory there is a Künneth formula: there exists a natural homomorphism of A^U-modules

$$U^*(K_1, L_1) \otimes_{\Omega_U} U^*(K_2, L_2) \to U^*(K_1 \times K_2 \,/\, K_1 \times L_2 \cup L_1 \times K_2),$$

which is an isomorphism for complexes without torsion, and the algebra

A^U acts on $U^*(K_1, L_1) \otimes_{\Omega_U} U^*(K_2, L_2)$ by means of the diagonal Δ indicated above.

To construct operations from A^U we shall use an analogue of the Chern characteristic classes (see ([7]), §§ 1, 2).

Lemma 1. There exist unique Chern characteristic classes

$$\sigma_k : k^0(K) \to U^{2k}(K),$$

having the following properties:

1. $\sigma_0 = 1$; if η is a U_1-bundle, then
$$\sigma_1(\eta) \in \mathrm{Map}(K, MU_1).$$
2. The Whitney formula
$$\sigma(\xi \oplus \eta) = \sigma(\xi)\sigma(\eta), \qquad \sigma = \sum_i \sigma_i.$$
3. For any U_1-bundles ξ, η, the class $\sigma_1(\xi \oplus \eta)$ is represented in the form
$$\sigma_1(\xi) + \sigma_1(\eta) + a(\sigma_1(\xi)\sigma_1(\eta))a \in A^U_2.$$
4. $$\sigma_1(\lambda^{-1}\xi) = a_{n-1}\sigma_n(\xi), \qquad \xi \in \mathrm{Map}(K, BU_n), \quad a_{n-1} \in A^U.$$

As usual, the classes σ_k generate classes σ_ω, where $\omega = (k_1, \dots, k_s)$ is an unordered partition of the number k into positive summands k_i (or $\omega = (0)$ and $k = 0$), and $\sigma_{(1,\dots,1)} = \sigma_i$;

$$\sigma_\omega(\xi \oplus \eta) = \sum_{\omega = (\omega_1, \omega_2)} \sigma_{\omega_1}(\xi)\sigma_{\omega_2}(\eta).$$

In U-theory there is a natural Thom isomorphism

$$\varphi_U : U^*(K) \to U^*(M\xi, P),$$

where ξ is a complex bundle over K, P is a point, and $M\xi$ is the Thom complex. For manifolds M^n we shall denote by $D\sigma_\omega(M^n)$ the dual (see [1]) bordism classes

$$D\sigma_\omega \in U_{n-2k}(M^n)$$

(normal).

Lemma 2. For closed quasicomplex manifolds, the bordism classes

$$D\sigma_\omega(M^n, \eta),$$

where η is the normal U-bundle to M^n, determine unique homomorphisms

$$\sigma_\omega^* : \Omega_U \to \Omega_U, \qquad \sigma_\omega^* \Omega_U^n \subset \Omega_U^{n-2k},$$

$\omega = (k_1, \dots, k_s)$, $\sum k_i = k$,

$$\sigma_\omega^*[M^n] = \varepsilon D\sigma_\omega(M^n, \eta), \qquad [M^n] \in \Omega_U^n,$$

$$\varepsilon : U_*(M^n) \to U_*(P)$$

is the natural homomorphism.

We note that for $n = 2k$, $\sigma_\omega^*(M^n)$ are characteristic numbers; all the homomorphisms σ_ω^* are easily computed for complex projective spaces:

$$\sigma_\omega^*(CP^n) = \lambda_\omega [CP^{n-k}], \qquad \lambda_k = k.$$

From Lemma 1 follows the Leibniz formula

$$\sigma_\omega^*(xy) = \sum_{(\omega_1, \omega_2) = \omega} \sigma_{\omega_1}^*(x) \sigma_{\omega_2}^*(y), \qquad xy \in \Omega_U.$$

With the help of Lemmas 1 and 2 one proves an important theorem which completely computes the ring

$$A^U = \sum_i A_i^U.$$

Theorem 2. 1) There exist unique operations

$$S_\omega : U^j(K, L) \to U^{j+2k}(K, L), \qquad \omega = (k_1, \dots, k_s), \quad \sum k_i = k,$$

having the following properties:
a) the operations S_ω commute with continuous mappings, the suspension isomorphism, and the homomorphism

$$\delta : U^{j-1}(L) \to U^j(K, L);$$

b) for any $\alpha, \beta \in U^*(K, L)$ the formula holds

$$S_\omega(\alpha\beta) = \sum_{\omega=(\omega_1,\omega_2)} S_{\omega_1}(\alpha) S_{\omega_2}(\beta);$$

c) if

$$\alpha \in \mathrm{Map}(K, MU_1) \subset U^2(K),$$

then

$$r^0 S_k(\alpha) = \alpha^{k+1}, \qquad S_\omega(\alpha) = 0, \quad \omega \neq (k);$$

d) the composition $S_{\omega_1} \circ S_{\omega_2}$ is a linear combination, in the ring A^U, of operations of the form S_ω;

e) if

$$K = CP_1^N \times \cdots \times CP_n^N,$$

where n, N are large, and

$$u_i \in U^2(CP_i^N)$$

are the elements dual to the submanifolds

$$CP_i^{N-1} \subset CP_i^N,$$

$u = u_1 \cdots u_n$, then the operations $S_\omega(u)$ are linearly independent for all ω, $\dim \omega < n$, and the linear $\mathbb{Z}$-space spanned by $S_\omega(u)$ is the ideal in the ring of symmetric polynomials in $u_1, \dots, u_n$ generated by the element u;

f) the Chern classes σ_ω are equal to

$$\varphi_U^{-1} S_\omega \varphi_U(1).$$

2) Every element $\gamma \in A_{2k}^U$ is represented uniquely in the form of a linear combination

$$\sum_{i \to \infty} \lambda_i x_i S_{\omega_i},$$

where x_i is some additive homogeneous basis of the ring Ω_U and

$$\dim(x_i S_\omega) = 2k,$$

λ_i are integers.

The ring A^U is a graded topological ring with topological basis $x_i S_\omega$, and the series

$$\sum \lambda_i x_i S_{\omega_i}$$

converge if $\dim \omega_i \to \infty$ as $i \to \infty$.

The equality $A^U = (\Omega_U \circ S)$ holds, where S is the ring generated by all S_ω.

3) The following commutation relation holds for the subrings $\Omega_U \subset A^U$ and $S \subset A^U$:

$$S_\omega \circ x = \sum_{\omega=(\omega_1,\omega_2)} \sigma^*_{\omega_1}(x)\, S_{\omega_2}.$$

Adem' s formulas in the ring A^U follow completely from items c)–e) and 3) by the usual Cartan method ([6]).

Example. For $K = S^0,\ L = P$ we have: the module $M_U = U^*(P)$ is given by one generator $t \in M^0_U$ and the relation $S_\omega(t) = 0,\ \dim\omega > 0$.

Theorem 3. If there is a mapping $A^U \xrightarrow{d} A^U$, where $d(1) = S_\omega$, then the corresponding mapping

$$d^* : \mathrm{Hom}_{A^U}(A^U, M_U) \to \mathrm{Hom}_{A^U}(A^U, M_U)$$

will coincide with $\sigma^*_\omega : \Omega_U \to \Omega_U$, where Ω_U is naturally isomorphic to the group $\pi^S_*(MU)$, $\pi^S_*(MU) = \mathrm{Hom}_{A^U}(A^U; M_U)$.

III. Let Q_p be the ring of rational numbers with denominator not divisible by p. To study the p-component of the groups $\pi^S_*(K)$ it suffices to study the ring $A^U \otimes_Z Q_p$ and the modules $U^*(K) \otimes_Z Q_p$. Let C be the class of finite groups of orders relatively prime to p. As follows from ([9–11]), the spectrum MU is C-homotopically equivalent to the direct sum $\sum_\omega M_\omega$, where ω are non-p-adic partitions $(k_1, \dots, k_m)$, and the A-module $H^*(M_\omega, \mathbb{Z}_p)$ is $A/A\bar{B}$, B being generated by elements $e'_r \in A^{2n-1}$ (recently Brown and Peterson in ([5]) constructed the spectrum M_ω). Let $X = M_\omega$ and $A^U_p = X^*(X) \otimes_Z Q_p$ be a graded ring over Q_p. The following simple theorem holds.

Theorem 4. A. The ring $A^U \otimes_Z Q_p$ is isomorphic to $GL(A^U_p)$, where $GL(A^U_p)$ is the graded ring of matrices over A^U_p of the form $(a_{\omega_i,\omega_j}) \in GL(A^U_p)$, $a_{\omega_i,\omega_j} \in A^U_p$,

$$n = \dim\omega_i - \dim\omega_j + \dim a_{\omega_i,\omega_j}$$

is the dimension of the matrix.

B. In the ring A^U_p lies the subring

$$\Omega_U(p) = \mathbb{Z}[x_1, \dots, x_i, \dots] \subset \Omega_U \otimes Q_p, \qquad \dim x_i = 2p^i - 2,$$

a projector

$$\pi_p : \Omega_U \otimes Q_p \to \Omega_U(p)$$

is defined,

$$\pi_p(xy) = \pi_p(x)\pi_p(y);$$

the generators x_i are such that

$$\sigma^*_{(2p^i-2)} x_i = p$$

and, for all ω, $\dim \omega = 2p^i - 2$,

$$\sigma_\omega^* x_i = 0 \pmod p$$

(the remaining choice of the x_i is arbitrary, but they are fixed).

C. The algebra S_p consists of all elements α of the algebra $S \otimes Q_p$, and they are identified if

$$\pi_p \alpha_1^* \pi_p = \pi_p \alpha_2^* \pi_p$$

on $\Omega_U(p)$, where $\alpha^* : \Omega_U \to \Omega_U$ are the homomorphisms of Lemma 2.

D. $X^*(P) = M_U(p)$, where $M_U(p)$ has a generator f and is given by the relation

$$S_p(t) = 0$$

($M_U, M_U(p)$ are the corresponding modules for the sphere).

Although the description of the ring A_p^U in Theorem 4 is algebraically complete, it is inconvenient for computations, and it would be better to find direct composition formulas.

IV. We now turn to k-theory, where $k_{2n} = BU^{(2n)}$ and $k_0 = BU \times Z$, $k_2 = BU$, $k_4 = BSU$, $\Omega^{2n} k_{2n} = BU \times Z$ (Bott), and embeddings $x : k_{2n} \to k_{2n-2}$ are defined for all n. For k-theory so defined we have that the functors $k^i(K, L)$ are isomorphic to the ordinary ones for $i \leq 0$, and for $i > 0$, $k^{2i}(K, L)$ consists of all elements of K^0 of filtration $\geq 2i$ for complexes without torsion. There is a Bott operator

$$x : k^j(K, L) \to k^{j-2}(K, L),$$

representing an element of dimension (-2) from the Steenrod ring A^k. We use two approaches to computing the ring A^k: the second of them uses Adams operations

$$\Psi^k : K^0 \to K^0$$

and the Bott operator

$$x : k^j \to k^{j-2},$$

while the first is based on the immersion $k^2 \to U^2$ (7). Both are incomplete.

1st method: we use the "corrected" Conner-Floyd operators

$$\lambda_{-1} : U^j \to k^j, \quad |j| < \infty$$

and

$$\sigma_1 : k^2 \to U^2,$$

which give a splitting of theories

cohomology $U^2 = k^2 + \cdots$, $\sigma_1 \circ \lambda_{-1} : U^2 \to U^2$ is the projector (7); we have an embedding $x^N A_j^k \to A_{j-2N}^U$, N large. Note that $x^N \to a_N$ (see Lemma 1) and $\sigma_1 \lambda_{-1}(\xi) = a_{N-1} \sigma_N(\xi)$, where ξ is a U_N-bundle, x is the Bott operator.

The 2nd method of defining operations in k-theory: the Adams operations Ψ^k (see (3)) do not exist in stable k-theory, since $\Psi^k \circ x = kx \circ \Psi^k$, but there do exist unstable operations $k^n \Psi^k : k^{2n}(K, L) \to k^{2n}(K, L)$. Let n be large. We choose a large number m and form a linear combination $\sum_k \lambda_k^{(n)} k^n \Psi^k = a_n$ such that the mappings of homotopy groups $a_{n*}^{(j)} : \pi_{2n+2j}(BU^{(2n)}) \to \pi_{2n+2j}(BU^{2n})$, for $j \leq m$, do not depend on n in the sense that for all $N > n$ one can find numbers $\lambda_k^{(N)}$ such that the homomorphisms

$$a_{n*}^{(j)} = \sum_k \lambda^{(n)} k^{n+j}$$

for $j \leq m$ coincide with

$$a_{N*}^{(j)} = \sum_k \lambda_k^{(N)} k^{N+j}.$$

Such a sequence $a = (a_n)$, where $a_{n*}^{(j)}$ does not depend on n for $j \leq m(n) \to \infty$, we shall call an operation in k-theory. If $a_{n*}^{(j)} = 0$ for $j < q$, then $a = x^q b$, where x is the Bott operator and $b : k^l \to k^{l+2q}$. We denote the ring of operations so constructed (together with x) by A_Ψ^k. If one takes the sphere module M_k^Ψ with one generator t over A_Ψ^k, and such that $bt = 0$ for all positive dimensions, then the following fact holds: $\mathrm{Ext}_{A_\Psi^k}^{1,2i}(M_k, M_k)$ is a cyclic group of order d_i, where d_i is the greatest common divisor of all the numbers $k^n(k^i - 1)$, over all k and for large n.

V. We indicate here some of the simplest results of computations.

Theorem 5. 1) The group $\mathrm{Ext}_{A_\Psi^k}^{1,2i}(M_k, M_k)$ is $\mathbb{Z}_{d_i}$, where d_i is the greatest common divisor of the numbers $k^n(k^i - 1)$ over all k, $n \to \infty$.

2) The group $\mathrm{Ext}_{AU}^{1,4i+2}(M_U, M_U)$ is $\mathbb{Z}_2$.

3) The groups $\mathrm{Ext}_{AU}^{1,4i}(M_U, M_U)$ are $\mathbb{Z}_{d_i/a_i}$, where $a_{2q} = 1$, $a_{2q+1} = 2$, $d_i/2$ is the denominator of the fraction $B_i/2i$, B_i is a Bernoulli number.

4) $d_i/\mathrm{Ext}_{AU}^{1,2i}(M_U, M_U) = 0$, $i \neq 4k-1$; $d_3/E_3^{1,8k+6} \neq 0$, where d_i are the differentials of the Adams spectral sequence (E_r, d_r), $k \geq 0$.

5) The homomorphism $q = \mathrm{Ext}^1 \circ J : \pi_{2i-1}(SO) \to \pi_{N+2i-1}(S)^N \to \mathrm{Ext}_{AU}^{1,2i}$ coincides exactly with the Hopf-Milnor-Kervaire invariant (8).

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Note: Figure translations are in progress. See original paper for figures.