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# ON MAPPINGS INTO THE SPHERE OF A BANACH SPACE

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## Abstract

## Full Text

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**MATHEMATICS**

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# ON MAPPINGS INTO THE SPHERE OF A BANACH SPACE

*(Presented by Academician P. S. Aleksandrov on 11 XI 1966)*

In the well-known paper of Leray and Schauder ([1]), for mappings of the form $I - F$, where $F$ is a completely continuous mapping, the degree of the mapping is defined with respect to points not lying on the image of the boundary. It turns out that the same property is possessed by the class of mappings in a Hilbert space of the form $I - K - F$, where $K$ is a **strictly contractive** mapping, i.e. $\|K(x) - K(y)\| \leq q\|x - y\|$ and the constant $q < 1$. In the main, this paper considers mappings for which it is possible to define the degree of the mapping mod 2 (see § 1). Several fixed-point theorems are given, whose proof is based on the fact that, under certain conditions, the degree of the mapping remains unchanged under a continuous deformation (see § 2).

In particular, a continuous mapping $f$ of a closed ball $W$ of a Banach space $E$ into $E$ has a fixed point if $f = K + F$, where $K$ is a strictly contractive mapping, $F$ is a completely continuous mapping, and $f(W) \subseteq W$.

1. Let $E$ be a real Banach space; $S(E)$ the unit sphere of $E$; $T(E)$ the set of all finite-dimensional subspaces of $E$. The set $T(E)$ is directed by inclusion. By $R(E)$ we denote the set of all oriented finite-dimensional subspaces of $E$. By $|R|$ we denote the corresponding subspace without orientation; we shall write $R_1 \subseteq R_2$ if $|R_1| \subseteq |R_2|$. The linking coefficient in $R \in R(E)$ of two cycles $z_1$ and $z_2$, $\vartheta_R(z_1, z_2)$, is set equal to $z_1 \times w$, where $w$ is a chain in $R$ bounding $z_2$. In particular, if $a$ is a point of $R$, $z$ is a cycle in $R$, and $\dim z = \dim R - 1$, then

$$\vartheta_R(z, a) = z \times_R (a, \infty).$$

Let $T \in T(E)$. By $D(T)$ we denote the complement of the set of points $x$ that can be represented in the form $x = y + z$, where $y \in T$ and $\|z\| < \|y\|$. The set $D(T) \cap S(E)$ is the carrier of a cycle of dimension (defect) $\dim T + 1$, linked with $T \cap S(E)$ on $S(E)$.

Let $\Omega$ be a bounded open set in $E$; $\overline{\Omega}$ its closure, $\dot{\Omega} = \overline{\Omega} \setminus \Omega$; and let $f$ be a continuous mapping of $\overline{\Omega}$ into $E$. Let $T_0 \in T(E)$. We define a **class of**

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**mappings** $A(T_0)$ as follows: $f \in A(T_0)$ if, for all $T \supseteq T_0$,

$$f(\dot{\Omega} \cap T) \cap DT = \emptyset.$$

Thus, if $E$ is a Hilbert space, then $f \in A(T_0)$ if, for any $T \supseteq T_0$, the orthogonal projection of the image of the boundary of the set $\Omega \cap T$ does not contain the point $O$, the origin.

**Theorem 1.** *Let $\Omega$ be a bounded open set in a Hilbert space $H$; let $f$ be a continuous mapping $\overline{\Omega} \to H$, and suppose $f$ can be represented in the form*

$$f(x) = x - L(x) - F(x),$$

*where $F$ is a completely continuous mapping and*

$$\|L(x) - L(y)\| \leq \|x - y\|.$$

*If $f(\dot{\Omega})$ lies at a positive distance from $O$, the origin, then there exists $T_0 \in T(H)$ such that $f \in A(T_0)$.*

If $f \in A(T_0)$, then the **degree of the mapping** $\bar{c}(f)$ of the mapping $f$ with respect to the point $O$ mod 2 is defined, as follows. For $T \supseteq T_0$, the degree of the mapping is defined

$c(f, T)$ of the mapping $f$ on $T \cap \Omega$. If $E$ is a Hilbert space and $T \supseteq T_0$, then $p_T f(T \cap \dot{\Omega})$ does not contain the point $O$, the origin; by $p_T$ we denote the orthogonal projection onto $T$. Consequently, the degree of the mapping $p_T f$, considered on the set $T \cap \Omega$, is defined. It is taken to be $c(f, T)$.

In the general case $c(f, T)$ is defined as follows. Take $\varepsilon = +1$ or $-1$, and to each $R \in R(E)$ assign such an oriented sphere $S(R)$, $|S(R)| = |R| \cap S(E)$, that $\vartheta_R(S(R), O) = \varepsilon$. This is equivalent to prescribing a definite orientation of $E$. Denote by $z(R)$ the boundary $\Omega \cap |R|$ oriented into a cycle in accordance with the orientation of $R$, i.e. $\vartheta_R(z(R), a) = \varepsilon$, if $a$ lies inside $\Omega \cap |R|$. Let $|R| = T$. By $p$ denote the projection onto $S(E)$ from $O$, and put $h = pf$; $h$ maps $\dot{\Omega}$ into $S(E)$. Let $a$ be the distance between $h(\dot{\Omega} \cap T)$ and $D(T)$. Choose $R_1 \supset R$ so that there exists an $a/2$-shift $\varphi$ of the compact $h(\dot{\Omega} \cap T)$ in $S(R_1)$. By $D_{R_1}(S(R))$ we denote such an integral cycle, lying in $D(T) \cap |S(R_1)|$, that

$$\vartheta_{S(R_1)}(D_{R_1}(S(R)), S(R)) = \varepsilon.$$

The number

$$(-1)^{\dim T} \vartheta_{S(R_1)}(D_{R_1}(S(R)), \varphi h z(R))$$

does not depend on the choice of $\varphi$ and $R_1$, and we take this value to be $c(f, T)$. The residue of $c(f, T)$ modulo 2 is denoted by $\bar{c}(f, T)$. It turns out that if $f \in A(T_0)$, then $\bar{c}(f, T)$ does not depend on $T$, if $T \supseteq T_0$, and this value is

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taken to be $\bar{c}(f)$. Stabilization of $c(f,T)$ for $T \supseteq T_0$ need not occur, as is shown by the example of the mapping $f(x) = -x$, considered on the ball with center at $O$. However, for certain classes of mappings stabilization of $c(f,T)$ does occur. The corresponding value of the degree of the mapping is denoted by $c(f)$.

**Theorem 2.** *Let $\Omega$ be a bounded open set in a Hilbert space $H$; let $f$ be a continuous mapping $\bar{\Omega} \to H$, and suppose that $f$ can be represented in the form $f = I - K - F$, where $K$ is a strictly contractive mapping and $F$ is a completely continuous mapping. If $f(\dot{\Omega})$ does not contain the origin, then there exist $T_0 \in T(H)$ and $\alpha > 0$ such that the mapping*

$$g_t = I - p_{T_0}K - (1-t)[K - p_{T_0}K] - F$$

*has the following property:*

$$\|p_T g_t(\dot{\Omega} \cap T)\| > \alpha, \quad 0 \le t \le 1,$$

*if $T \supseteq T_0$, $p_T$ is the orthogonal projection onto $T$. Consequently, $f = g_0$ has the same degree of mapping as the mapping $g_1 = I - F_1$, where $F_1 = F + p_{T_0}K$ is a completely continuous mapping.*

**2.** Let $\Omega$ be a bounded open set in $E \times K$, where $K$ is a segment of the number axis, and let $\Omega(k)$ be the set of those points $x \in E$ such that $(x,k) \in \Omega$. Let $f$ be a continuous mapping of $\bar{\Omega}$ into $E$, with $O \notin f(\dot{\Omega})$, and suppose that for every $k \in K$ the mapping $f$, considered on $\Omega(k)$, belongs to the class $A(T(k))$, $T(k) \in T(E)$. Then $\bar{c}(f,k)$ is defined.

If all $T(k)$ are bounded above, i.e. if there exists $T_0 \in T(E)$ such that for all $k \in K$, $T(k) \subseteq T_0$, then $\bar{c}(f,k)$ preserves a constant value.

But if the $T(k)$ are not bounded above, then, generally speaking, $\bar{c}(f,k)$ may undergo a discontinuity under a continuous change of $k$, as is shown by the example given in § 3.

It follows from Theorem 2 that if $f_t = I - K_t - F_t$, where $K_t$ is a strictly contractive mapping, $F_t$ is a completely continuous mapping, and $f(\dot{\Omega}(t))$ does not contain the origin of the Hilbert space, $0 \le t \le 1$, then $c(f_t)$ is a constant quantity.

**3.** The following simple example shows that from $\bar{c}(f) \neq 0$ on $\dot{\Omega}$ it does not follow that $O \in f(\bar{\Omega})$, and that under a continuous deformation $f_t$ the quantity $\bar{c}(f_t)$ may undergo a jump.

Let $H$ be a Hilbert separable space; let $U$ be the open ball $\|x\| < 1$, and let $S$ be the boundary of $U$; $\Omega = U \times [0,1]$. Let $x = (x_1, x_2, ...) \in S$. Put

$$g_t(x) = (3t, x_1, x_2, ...), \quad 0 \le t \le 1.$$

On $S$ we have $g_t(x) \neq x$,

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$0 \leq t \leq 1$. We define $f(x,t)$ on $S$ as follows:

$$f(x,t) = \frac{g_t(x) - x}{\|g_t(x) - x\|}.$$

Let $\xi = (3, 0, 0, ...)$. Put $f(0,t) = \xi$ and

$$f(x,t) = \xi(1 - \|x\|) + \|x\| f(x/\|x\|, t),$$

if $0 \leq t \leq 1$, $0 \leq \|x\| < 1$. It can be shown that $f(x,t) \in A(T(t))$, $T(t) \in T(H)$, $0 \leq t \leq 1$. For $t = 0$ we have $(f(x,0), x) \neq 0$, $x \in S$. Hence it follows that $\bar{c}(f,0) = 1$. From the fact that for $t = 1$, $f(S,1) \subseteq O_1(\bar{\xi})$, it follows that $\bar{c}(f,1) = 0$. Although $\bar{c}(f,0)$ on $\bar{U}$ is equal to 1, $f(\bar{U},0)$ does not contain $O$.

4. In this section we give several fixed-point theorems for certain classes of contractive (in one sense or another) mappings. In this section $F$ always denotes a completely continuous mapping. For brevity, by $C_i$, $1 \leq i \leq 5$, we shall denote the following classes of continuous mappings: 1) $C_1$—the class of nonexpansive mappings, i.e. $\|f(x) - f(y)\| \leq \|x - y\|$; 2) $C_2$—the class of mappings representable in the form $f = K + F$, where $K$ is a strictly contractive mapping; 3) $C_3(q)$, $q \geq 0$, $f \in C_3(q)$, if there exists a compact set $\Phi$ (depending on $f$) such that $\rho(f(x), \Phi) \leq q\rho(x, \Phi)$; 4) $C_4$—the class of mappings representable in the form $f = h + F$, where $h$ is a compacting mapping. A mapping $h$, defined on $\bar{\Omega}$ and mapping $\bar{\Omega}$ into a bounded set of a real Banach space $E$, will be called compacting on $\bar{\Omega}$ if there exist $q < 1$ and a sequence $T_n \in T(E)$ such that for any point $x \in \bar{\Omega}$ and any $\alpha > 0$ there exists an $m = m(x, \alpha)$ such that $h(W_\alpha(x) \cap \Omega) \in O_{\alpha q}(T_n)$ for $n > m$, where $W_\alpha(x)$ is the ball of radius $\alpha$ with center at $x$.

Browder, in a series of works, investigated in detail the properties of nonexpansive mappings. In particular, he proved ([2]) that in a Hilbert space a nonexpansive mapping $f$, defined on a closed ball $V$, has a fixed point in $V$, if $f(S) \subseteq V$. The same assertion was proved by him for uniformly convex Banach spaces ([3]).

**Theorem 3.** Let $W$ be a closed ball of a real Banach space $E$; $f : W \to E$ and $f(W) \subseteq W$. The mapping $f$ has a fixed point if $f \in C_2$ or $f \in C_3(q)$ and $q < 1$.

**Theorem 4.** Let $H$ be a real Hilbert space; $V$ a closed ball with center at the origin; $S$ its boundary. Let $f$ be a mapping of $V$ into $H$ such that $f(S) \subseteq V$ or $(f(x), x) \neq (x, x)$, if $x \in S$.

The mapping $f$ has a fixed point in $V$, if $f$ belongs to one of the classes $C_1$, $C_2$, $C_3(q)$, where $q < 1$, $C_4$. If, however, $f \in C_3(1)$, or if $f$ is representable in the form $f = \varphi + F$, $\varphi \in C_1$, then

$$\inf_{x \in V} \|x - f(x)\| = 0.$$

The proof is carried out according to the following scheme. If the mapping $f$

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has no fixed point, then we can consider the mapping $\varphi_t(x) : S \to S$, defined as

$$\frac{f(\tau x) - \tau x}{\|f(\tau x) - \tau x\|}, \quad \tau = \frac{1}{t}, \quad 0 < t \leq 1.$$

If we assume that the assertion of the theorem for the classes indicated in Theorem 4 does not hold, then one can prove that for all $t$, $t \in [\beta, 1]$, where $\beta > 0$, the mapping $\varphi_t \in A(T(t))$ and all $T(t)$ are bounded. This entails the constancy of the degree of the mapping $\bar{c}(\varphi_t)$; from the fact that $f(0) \neq 0$, it follows that for sufficiently small $\beta$, $\bar{c}(\varphi_\beta) = 0$. On the other hand, if $f(x) \neq x$ on $S$, then $\bar{c}(\varphi_1) = 1$ under the assumptions made about $f$ on $S$.

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[3] F. E. Browder, ibid., 54, 1041 (1965).

*Note: Figure translations are in progress. See original paper for figures.*

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