ON THE THEORY OF CONSTRUCTING MODELS OF CONTINUOUS MEDIA

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Abstract

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MECHANICS

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ON THE THEORY OF CONSTRUCTING MODELS OF CONTINUOUS MEDIA

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We shall consider material media in a pseudo-Riemannian space G of index 3, referred to the observer' s coordinate system x^i with metric tensor $g = \|g_{ik}\|$. Introduce in the space G a nonholonomic coordinate system $\tilde{x}^\alpha$, consisting of orthonormal frames $\{e_\alpha\}$ with metric tensor $\tilde{g}_{\alpha\beta}$, assigned to each point of the space G, and also an accompanying coordinate system ξ^i such that the functions $x^i = x^i(\xi^1, \xi^2, \xi^3, \xi^4)$ give the law of motion of individual points of the medium. Components of tensors given in the system x^i will be denoted by Latin indices, and components of tensors given in the system $\tilde{x}^\alpha$ by Greek indices. The relation between the holonomic coordinate system x^i and the nonholonomic system $\tilde{x}^\alpha$ is effected by the formulas

$$\partial_i = \Omega_i^\alpha e_\alpha, \qquad e_\alpha = \Omega_\alpha^i \partial_i, \qquad \Omega_\alpha^i = g^{ip} \tilde{g}_{\alpha\beta} \Omega_p^\beta, \tag{1}$$

where ∂_i are the basis vectors of the system x^i, and $\|\Omega_i^\alpha\|$ is the metric Lamé matrix. The components Ω_i^α are assumed to be sufficiently smooth functions of the coordinates x^i. Parallel displacement in the systems $x^i, \tilde{x}^\alpha$ is determined by the relations

$$\partial e_\alpha / \partial x^j = \Delta_{j,\alpha}^\beta e_\beta, \tag{2}$$

where

$$\Gamma_{ij}^k = \frac{1}{2} g^{kp} \left(\partial g_{ip}/\partial x^j + \partial g_{jp}/\partial x^i - \partial g_{ij}/\partial x^p \right)$$

are the Christoffel symbols, and

$$\Delta_{j,\alpha}^\beta = \Omega_\alpha^i \Omega_k^\beta \left(\Gamma_{ij}^k - \Omega_\rho^k \partial \Omega_i^\rho / \partial x^j \right)$$

are the Ricci symbols. From (1), (2) we have

$$\nabla_k \Omega_i^\alpha = \partial \Omega_i^\alpha / \partial x^k - \Gamma_{ik}^p \Omega_p^\alpha - \Delta_{k,\beta}^\alpha \Omega_i^\beta = 0.$$

Let γ_α $(\alpha = 1, 2, 3, 4)$ be Dirac matrices, by definition satisfying the equation $\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 2\tilde{g}_{\alpha\beta}I$, where I is the unit matrix. A four-component object $\Psi = \{\psi^\alpha\}$, defined up to sign, which transforms under the representation S of the group $L = \|l_\alpha^\beta\|$ of orthogonal transformations of the coordinates $\tilde{x}^\alpha$, recoverable from the equation $\gamma_\alpha = l_\alpha^\beta S\gamma_\beta S^{-1}$, is called a spinor of first rank in the space G. The covariant derivative for spinors is defined by the formula $\nabla_k\psi = \partial\psi/\partial x^k - \Gamma_k\psi$, where Γ_k are determined through the Ricci symbols [1]

$$\Gamma_k = \frac{1}{4}\gamma_{\alpha\beta}\Delta_k^{\alpha\beta}, \qquad \gamma_{\alpha\beta} = \frac{1}{2}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha). \tag{3}$$

Spinors of higher ranks and the covariant derivatives for them are defined analogously to how this is done for tensors.

We shall use the variational principle in the form [2,3]

$$\delta\int_V\left(\frac{1}{2\varkappa}R + \Lambda_m\right)d\tau + \delta W^* + \delta W = 0. \tag{4}$$

Here V is an arbitrary four-dimensional volume in

$$d\tau = \sqrt{-g}\,dx^1dx^2dx^3dx^4$$

–the element of four-dimensional volume; $g = \det\|g_{ik}\|$; $\varkappa$ is the gravitational constant; R is the scalar curvature of the space G; Λ_m is the Lagrangian of matter; δW^* is a prescribed functional of the defining parameters of the medium; δW is a functional determined by the specification of Λ_m and δW^*. The introduction of δW^* is connected with consideration of irreversible processes [2].

Let us consider, within the framework of the general theory of relativity, a model of a medium for which the Lagrangian Λ_m depends on certain parameters μ^A and on their first n derivatives with respect to coordinates and time, on the quantities $x_p^i = \partial x^i/\partial\xi^p$ and on their derivatives with respect to coordinates and time, and also on the metric Lamé tensor Ω_i^α. Without loss of generality one may assume that Λ_m also depends on n derivatives of x_p^i. The Lagrangian Λ_m may also contain constant tensors, invariant throughout the entire space, nonvarying tensors $\gamma_\alpha, \tilde{g}_{\alpha\beta}$, etc.* The independent defining parameters μ^A may be tensors and spinors of any rank. The physical meaning of the parameters μ^A must be clarified from the equations of motion of the medium, or introduced a priori. As is known ([4,5]), specifying a spinor is equivalent to specifying a system of tensors of a special kind. However, the use of spinors in variational principles as generalized parameters is more convenient than the use of a system of equivalent tensors, in view of the presence of algebraic relations between the components of these tensors. Nevertheless, the use of equivalent tensors in the final formulation of the theory is useful for clarifying the physical meaning of the spinor equations.

Let us choose the functional δW^* in the form

$$\delta W^* = \int_V \left[Q_k \delta x^k + Q_A \delta\mu^A + \nabla_i \left(Q_k^j \delta x^k + \sum_{q=1}^{n} Q_k^{ij_1\cdots j_q} \nabla_{j_1\cdots j_q} \delta x^k + \sum_{q=0}^{n-1} Q_A^{ij_1\cdots j_q} \nabla_{j_1\cdots j_q} \delta\mu^A \right) \right] d\tau. \tag{5}$$

The coefficients Q entering into δW^* must be specified as functions of the defining parameters of the medium or else determined from prescribed equations. Using the definition of the complete variation $\delta\mu^A = \partial\mu^A + \delta x^k \nabla_k \mu^A$ ($\partial\mu^A = \mu^{A'} - \mu^A$ is the local variation), it is easy to obtain the formulas

$$\delta\nabla_{i_1\cdots i_n}\mu^A = \nabla_{i_1\cdots i_n}\delta\mu^A - \sum_{p=1}^{n} \nabla_{\{i_1\cdots i_p|}\delta x^k \nabla_{i_{p+1}\cdots i_n\}k}\mu^A +$$

$$+\delta x^k \left(\nabla_{ki_1\cdots i_n}\mu^A - \nabla_{i_1\cdots i_n k}\mu^A\right) + \sum_{p=1}^{n} \nabla_{i_1\cdots i_{p-1}} \left(\frac{\partial \nabla_{i_p} \nabla_{i_{p+1}\cdots i_n}\mu^A}{\partial\Gamma} \partial\Gamma \right),$$

$$\delta\nabla_{i_1\cdots i_n} x_p^j = \sum_{k=0}^{n} \nabla_{\{i_1\cdots i_k|s}\delta x^s \nabla_{i_{k+1}\cdots i_n\}} x_p^j - \sum_{k=1}^{n} \nabla_{\{i_1\cdots i_k|}\delta x^r \nabla_{i_{k+1}\cdots i_n\}r} x_p^j + \tag{6}$$

$$+\delta x^k \left(\nabla_{ki_1\cdots i_n} x_p^j - \nabla_{i_1\cdots i_n k} x_p^j\right) + \sum_{k=1}^{n} \nabla_{i_1\cdots i_{k-1}} \left(\frac{\partial \nabla_{i_k} \nabla_{i_{k+1}\cdots i_n} x_p^j}{\partial\Gamma} \partial\Gamma \right),$$

where $\nabla_{i_1\cdots i_n} = \nabla_{i_1}\cdots\nabla_{i_n}$; the expression $\partial\nabla_{i_k}\nabla_{i_{k+1}\cdots i_n}\mu^A/\partial\Gamma$ means that the derivative with respect to Γ is taken from the covariant derivative ∇_{i_k} of the tensor $\nabla_{i_{k+1}\cdots i_n}\mu^A$; the symbol $\{i_1\cdots i_k|i_{k+1}\cdots i_n\}$ denotes the sum of all possible permutations with the index numbers arranged in increasing order before and after the vertical bar; for example,

$\{i_1 i_2|i_3 i_4\} = (i_1 i_2 i_3 i_4) + (i_1 i_3 i_2 i_4) + (i_1 i_4 i_2 i_3) + (i_2 i_3 i_1 i_4) + (i_2 i_4 i_1 i_3) + (i_3 i_4 i_1 i_2)$;

Γ are the connection symbols of the object μ^A.

If μ^A is a tensor, then Γ are the Christoffel symbols Γ_{ij}^k,

$$\partial\Gamma_{ij}^k = B_{ij\alpha}^{kpq}\nabla_p\partial\Omega_q^\alpha,$$

$$B_{ij\alpha}^{kpq} = \frac{1}{2}\Omega_{r\alpha}\left[-g^{kp}(\delta_i^q\delta_j^r + \delta_j^q\delta_i^r) + g^{kr}(\delta_i^q\delta_j^p + \delta_j^q\delta_i^p) + g^{kq}(\delta_i^r\delta_j^p + \delta_j^r\delta_i^p)\right].$$

If μ^A is a spinor, then Γ are the coefficients Γ_k, defined according to (3). Using the definition of the coefficients Γ_k and the covariant equation $\nabla_k \Omega_i^\alpha = 0$, we obtain

$$\partial\Gamma_k = B_{k\alpha}^{lj} \nabla_l \partial\Omega_j^\alpha, \qquad B_{k\alpha}^{lj} = \frac{1}{8}\gamma^{pq}\left[\Omega_{p\alpha}(\delta_q^j \delta_k^l - \delta_k^j \delta_q^l) + \Omega_{q\alpha}(\delta_p^l \delta_k^j - \delta_k^l \delta_p^j) + \Omega_{k\alpha}(\delta_p^l \delta_q^j - \delta_q^l \delta_p^j)\right], \tag{7}$$

$$\gamma^{pq} = \frac{1}{2}(\gamma^p \gamma^q - \gamma^q \gamma^p), \qquad \gamma^p = \Omega^{p\alpha}\gamma_\alpha.$$

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* The case when Λ_m depends on the first two derivatives of μ^A was considered in ([6,7]). In ([7]) the case is considered when Λ_m depends on derivatives of a spinor. However, the variational technique used in the present work makes it possible to consider only Lagrangians whose spinor terms are invariant with respect to the choice of the system $\tilde{x}^\alpha$.

Carrying out the variation in equation (4) with δW^*, determined according to (5) taking formulas (6) into account, we obtain

$$\int_V \Bigg\{ \left[\frac{\partial\Lambda_m}{\partial\mu^A} + \nabla_i(G_A^i + Q_A^i) + Q_A\right]\delta\mu^A + \Bigg[\nabla_i(P_k^i + Q_k^i) + Q_k - \frac{1}{2\varkappa}\nabla_k R +$$
$$+ \sum_{q=1}^{n} \frac{\partial\Lambda_m}{\partial\nabla_{i_1\ldots i_q}\mu^A}\left(\nabla_{ki_1\ldots i_q}\mu^A - \nabla_{i_1\ldots i_q k}\mu^A\right)\delta x^k +$$
$$+ \sum_{q=1}^{n} \frac{\partial\Lambda_m}{\partial\nabla_{i_1\ldots i_q}x_p^j}\left(\nabla_{ki_1\ldots i_q}x_p^j - \nabla_{i_1\ldots i_q k}x_p^j\right)\delta x^k + \left[-\frac{1}{\varkappa}\left(R^{ik} - \frac{1}{2}g^{ik}R\right)\Omega_{ia} +\right.$$
$$\left. + \frac{\partial\Lambda_m}{\partial\Omega_k^a} + \Lambda_m\Omega_a^k + \nabla_i N_a^{ik}\right]\partial\Omega_k^a - \nabla_i\Bigg\{P_k^i\delta x^k + \sum_{q=1}^{n-1} P_k^{ij_1\ldots j_q}\nabla_{j_1\ldots j_q}\delta x^k +$$
$$+ \overline{G}_k^{sij_1\ldots j_{n-1}} x_s^j \nabla_{j_1\ldots j_{n-1}j}\delta x^k + \sum_{q=0}^{n-1} G_A^{ij_1\ldots j_q}\nabla_{j_1\ldots j_q}\delta\mu^A +$$
$$+ N_a^{ik}\partial\Omega_k^a - \frac{1}{2\varkappa}(g^{ki}\delta_s^j - g^{ik}\delta_s^j)\partial\Gamma_{kj}^s +$$
$$+ \sum_{q=0}^{n-2}\sum_{r=1}^{n-q-1}\left[\Phi_A^{ij_1\ldots j_q p_1\ldots p_r}\nabla_{j_1\ldots j_q}\left(\frac{\partial\nabla_{p_1}\nabla_{p_2\ldots p_r}\mu^A}{\partial\Gamma}\delta\Gamma\right) +\right.$$
$$\left. + \Phi_k^{sij_1\ldots j_q p_1\ldots p_r}\nabla_{j_1\ldots j_q}\left(\frac{\partial\nabla_{p_1}\nabla_{p_2\ldots p_r}x_s^k}{\partial\Gamma}\delta\Gamma\right)\right]\Bigg\}\Bigg\} d\tau + \delta W = 0. \tag{8}$$

Here it is denoted

$$P^i_k = \sum_{p=1}^{n}\sum_{r=0}^{p-1}(-1)^r \nabla_{i_r \dots i_1}\eta^{i_1\dots i_r i}_{(p)k} - \left[\frac{\partial \Lambda_m}{\partial x^k_q} + \nabla_s \overline{G}^{qs}_k\right] x^i_q - \left(\Lambda_m + \frac{1}{2\varkappa}R\right)\delta^i_k - Q^i_k,$$

$$P^{ij_1\dots j_q}_k = \sum_{p=q+1}^{n}\sum_{r=0}^{p-q-1}(-1)^r \nabla_{i_r\dots i_1}\eta^{i_1\dots i_r i j_1 \dots j_q}_{(p)k} +$$

$$+ \sum_{p=q-1}^{n-1} g^{a j_1 \dots j_{q-1}}_{i_1\dots i_{q-1}}\Big|^{ka\dots k_p}_{i_q\dots i_p} \overline{G}^{r i_1 \dots i_p}_k \nabla_{k_q \dots k_p} x^{j_q}_r - Q^{ij_1\dots j_q}_k,$$

$$G^{ij_1\dots j_q}_A = \sum_{p=0}^{n-q-1}(-1)^{p+1}\nabla_{i_p\dots i_1}\frac{\partial \Lambda_m}{\partial \nabla_{i_1\dots i_p i j_1 \dots j_q}\mu^A} - Q^{ij_1\dots j_q}_A,$$

$$\overline{G}^{sij_1\dots j_q}_k = \sum_{p=0}^{n-q-1}(-1)^{p+1}\nabla_{i_p\dots i_1}\frac{\partial \Lambda_m}{\partial \nabla_{i_1\dots i_p i j_1\dots j_q} x^k_s},$$

$$N^{ik}_a = \sum_{q=1}^{n}\sum_{l=1}^{q}(-1)^l \nabla_{i_{l-1}\dots i_1}\frac{\partial \Lambda_m}{\partial \nabla_{i_1\dots i_q}\mu^A}\frac{\partial \nabla_{i_l}\nabla_{i_{l+1}\dots i_q}\mu^A}{\partial \Gamma}B^{ik}_a + (\mu^A \to x^j_p),$$

$$\Phi^{ij_1\dots j_q p_1 \dots p_r}_A = \sum_{m=0}^{n-q-r-1}(-1)^{m+1}\nabla_{i_m\dots i_1}\frac{\partial \Lambda_m}{\partial \nabla_{i_1\dots i_m i j_1 \dots j_q p_1 \dots p_r}\mu^A},$$

$$\Phi^{sij_1\dots j_q p_1 \dots p_r}_k = \sum_{m=1}^{n-q-r-1}(-1)^{m+1}\nabla_{i_m\dots i_1}\frac{\partial \Lambda_m}{\partial \nabla_{i_1\dots i_m i j_1\dots j_q p_1\dots p_r} x^k_s},$$

$$\eta^{i_1\dots i_q}_{(p)k} = \left[\frac{\partial \Lambda_m}{\partial \nabla_{j_1\dots j_p}\mu^A}\nabla_{i_{q+1}\dots i_p k}\mu^A + \frac{\partial \Lambda_m}{\partial \nabla_{j_1\dots j_p} x^r_s}\nabla_{i_{q+1}\dots i_p k} x^r_s\right] g^{i_1\dots i_q}_{j_1 \dots j_q}\Bigg|^{i_{q+1}\dots i_p}_{j_{q+1}\dots j_p},$$

$$g^{i_1\dots i_q}_{j_1\dots j_q}\Bigg|^{i_{q+1}\dots i_p}_{j_{q+1}\dots j_p} = \delta^{i_1}_{\{j_1}\cdots\delta^{i_q}_{j_q}\delta^{i_{q+1}}_{j_{q+1}}\cdots\delta^{i_p}_{j_p\}}. \tag{9}$$

In formulas (6), (8), (9) the indices with zero number i_0, p_0, j_0 are omitted, and it is assumed that $\nabla_{i_0} = \nabla_{p_0} = \nabla_{j_0} = 1$.

Assuming the independent variations $\delta x^i, \delta\mu^A, \delta\Omega^\alpha_k$ to be zero on the boundary Σ of the volume V, from (8) we obtain

$$\nabla_i P^i_k = -(Q_k + \nabla_i Q^i_k) - \frac{1}{2\varkappa}\nabla_k R - \sum_{q=1}^{n}\frac{\partial \Lambda_m}{\partial \nabla_{i_1\dots i_q}\mu^A}\left(\nabla_{k i_1 \dots i_q}\mu^A - \nabla_{i_1\dots i_q k}\mu^A\right) - \sum_{q=1}^{n}\frac{\partial \Lambda_m}{\partial \nabla_{i_1\dots i_q} x^p_s}\left(\nabla_{k i_1\dots i_q} x^p_s - \nabla_{i_1\dots} \right. \tag{10}$$

$$\frac{\partial \Lambda_m}{\partial \mu^A} + \nabla_i G^i_A = -(Q_A + \nabla_i Q^i_A), \tag{11}$$

$$\frac{1}{\varkappa}\left(R^{ik}-\frac{1}{2}g^{ik}R\right)\Omega_{i\alpha}=\frac{\partial\Lambda_m}{\partial\Omega^\alpha_k}+\Lambda_m\Omega^k_\alpha+\nabla_i N^{ik}_\alpha, \tag{12}$$

where R^{ik} is the Ricci tensor.

For variations of the parameters that are not zero on the boundary Σ, by virtue of the equations of motion (10)–(12) we obtain

$$\delta W=\int_\Sigma n_i\Bigg\{P^i_k\delta x^k+\sum_{q=1}^{n-1}P_k^{ij_1\cdots j_q}\nabla_{j_1\cdots j_q}\delta x^k+\overline{G}_k^{sij_1\cdots j_{n-1}}x^j_s\nabla_{j_1\cdots j_{n-1}j}\delta x^k+$$

$$+\sum_{q=1}^{n-1}G_A^{ij_1\cdots j_q}\nabla_{j_1\cdots j_q}\delta\mu^A+N^{ik}_\alpha\delta\Omega^\alpha_k-\frac{1}{2\varkappa}\left(g^{kj}\delta^i_s-g^{ik}\delta^j_s\right)\delta\Gamma^s_{kj}+$$

$$+\sum_{q=0}^{n-2}\sum_{r=1}^{n-q-1}\left[\Phi_A^{ij_1\cdots j_q p_1\cdots p_r}\nabla_{j_1\cdots j_q}\left(\frac{\partial\nabla_{p_1\cdots p_r}\mu^A}{\partial\Gamma}\delta\Gamma\right)+\Phi_k^{sij_1\cdots j_q p_1\cdots p_r}\nabla_{j_1\cdots j_q}\left(\frac{\partial\nabla_{p_1}\nabla_{p_2\cdots p_r}x^k_s}{\partial\Gamma}\delta\Gamma\right)\right]\Bigg\}d\sigma,$$

where n_i are the components of the unit normal vector to the surface Σ.

For given tensors Q, equations (10), (11), (12) form a closed system of equations completely describing the medium. Equations (12), contracted with Ω^α_p, give the Einstein equations, whose right-hand side is the energy-momentum tensor.

The totality of the tensors P, G, Φ, N entering into δW determines the surface interaction between a volume V mentally singled out in the medium and the entire medium. The physical meaning of these tensors is determined by the nature of the parameters μ^A. The tensor P^i_k may be regarded as the second energy-momentum tensor, which is a generalization of the surface-stress tensor in nonrelativistic mechanics. The definition of δW^* and of the energy-momentum complex may be regarded as the equations of state for the medium ([2]).

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Note: Figure translations are in progress. See original paper for figures.