HOMEOMORPHY OF QUASICONFORMAL MAPPINGS OF SPACE

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Abstract

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HOMEOMORPHY OF QUASICONFORMAL MAPPINGS OF SPACE

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As early as 1938, in one of his papers ([1]), M. A. Lavrent' ev stated two very interesting assertions about the behavior of quasiconformal mappings in three-dimensional Euclidean space. These were the first assertions in which the specific nature of the spatial case was perceived. The first of them—on the removability of singularities of lower dimension for a quasiconformal mapping of a ball—may by now be regarded as essentially investigated (see ([2–5])). Below we present a scheme of proof of the second assertion of M. A. Lavrent' ev.

If $F : E^3 \to E^3$ is a locally homeomorphic quasiconformal mapping of three-dimensional Euclidean space E^3 into itself, then F is a homeomorphism, and moreover onto all of E^3.

The central point here is, of course, the assertion of the homeomorphy of the mapping, for then the subsequent assertion, on its surjectivity, is proved almost trivially by the method of moduli both in the spatial and in the plane case, and therefore contains nothing specific to space, whereas the first assertion in the plane case clearly does not hold ($w = e^z$).

The proof is based on the following auxiliary assertions.

Lemma 1. Let I be an open interval in E^3, $K(x)$ a disk of radius $r(x) > 0$ with center at the point $x \in I$, whose plane is orthogonal to I; let L be some measurable subset of I. On the boundary of each disk $K(x)$ corresponding to a point $x \in L$, fix some point $a(x)$, and consider the family $\Gamma(x)$ of all curves in $K(x)$ joining the points x and $a(x)$. If $\Gamma = \bigcup_{x\in L} \Gamma(x)$, then for the modulus of the family Γ the following lower estimate holds

$$M(\Gamma) \geq c\,\frac{\operatorname{mes}_1 L}{\sup\limits_{x\in L} r(x)},$$

where c is a positive constant.

Lemma 2. Let (r, α, h) be a cylindrical coordinate system in E^3; I a finite open interval of the axis h; $P(\alpha_0)$ a rectangle in the half-plane $\alpha = \alpha_0$, one of whose sides coincides with I, and the other is equal to $r(\alpha_0) > 0$; let ω be some measurable subset of the half-interval $[0, 2\pi)$. On the side opposite I of each rectangle $P(\alpha)$, corresponding to an angle $\alpha \in \omega$, fix some point $x(\alpha)$, and consider the set $\Gamma(\alpha)$ of all curves in $P(\alpha)$ joining $x(\alpha)$ to I. If $\Gamma = \bigcup_{\alpha\in\omega} \Gamma(\alpha)$, then for the modulus of the family Γ the following lower estimate holds

$$M(\Gamma) \geq \frac{I \operatorname{mes}_1 \omega}{\pi^2 \sqrt{I^2 + \sup\limits_{\alpha\in\omega} r^2(\alpha)}},$$

where I is the length of the interval I.

Remark 1. If a locally homeomorphic mapping F of a domain D is homeomorphic on a compact set $K \subset D$, then it is also homeomorphic on some δ-neighborhood of K.

Remark 2. Let F be a locally homeomorphic mapping of a domain D^* onto a domain D; let M_1, M_2 be subsets of D, and let M_1^*, M_2^* be subsets of D^*, on each of which F is homeomorphic, with

$$F(M_1^*) = M_1, \qquad F(M_2^*) = M_2.$$

If $M_1^* \cap M_2^*$ is nonempty, and $M_1 \cap M_2$ is connected, then F maps $M_1^* \cup M_2^*$ homeomorphically onto $M_1 \cup M_2$.

We now describe the proof itself directly.

I. Let $F(E^3)$ be the image of E^3 under the mapping F, and let I be a certain open interval in $F(E^3)$, on one of the components of whose preimage (denote it by I^*) the mapping F is homeomorphic. The existence of such an interval I is ensured by the local homeomorphy of F. For each point $x_0 \in I$, in the plane $H(x_0)$ passing through x_0 orthogonally to I, we construct the maximal disk

$$K(x_0) = \{x \mid |x - x_0| < r(x_0),\ x \in H(x_0)\}$$

such that the mapping F is homeomorphic on the component of the preimage of $K(x_0)$ that intersects I^*. We shall call this component the distinguished one. Using Remark 1, it is easy to verify that:

1^0. For every $x \in I$, the distinguished component of the preimage of $K(x)$ goes to infinity (is unbounded in E^3).

2^0. The union of the distinguished components over all $x \in I$ is a domain $D^* \subset E^3$.

3^0. The mapping F is homeomorphic in D^*.

4^0. The union of the disks $K(x)$ over all $x \in I$ is a domain $D \subset E^3$, retracting onto I, and $D = F(D^*)$.

5^0. The function $r(x)$ is lower semicontinuous on I.

6^0. The set

$$M = \{x \mid x \in I,\ r(x) < +\infty\}$$

is measurable on I. (After 5^0 this is, of course, obvious. Below it will be shown that it even has linear measure zero on I.)

II. Now take such a point $x \in I$ for which $r(x) < +\infty$ (if such a point does not exist, then, as will be clear from what follows, the situation only becomes simpler). Then, by virtue of 1^0, in the disk $K(x)$ there is a sequence of points converging to some point of its boundary whose preimages go to infinity along the distinguished component corresponding to the disk $K(x)$. Any boundary point of $K(x)$ to which such a sequence converges clearly prevents the extension of the disk $K(x)$, and therefore will henceforth be called a special point.

Thus, on the boundary of any disk $K(x)$ of finite radius there is at least one special point.

III. It is easy to see that on the distinguished component of the preimage of any sequence of points from $K(x)$ converging to a special point, there is always a sequence going to infinity.

IV. Now, relying on Lemma 1, one can show that the set

$$M = \{x \mid x \in I,\ r(x) < +\infty\}$$

has linear measure zero on I.

V. Thus, for almost all $x \in I$, the corresponding disks $K(x)$ coincide with the whole plane $H(x)$.

Returning again to the beginning of the proof, we may take the initial interval I to have been chosen precisely in such a plane, and therefore unbounded in both directions. All constructions and results already obtained in items I-IV then obviously remain in force. Everywhere in what follows we retain the previous notation for the corresponding objects constructed already under the assumption that I is a straight line in E^3.

VI. We now want to enlarge the domain D, but in such a way that one of the components of the preimage of the enlarged domain is still mapped homeomorphically by F onto this enlarged domain. To this end, on the straight line I we fix an arbitrary finite open interval $I_0 \subset I$. We shall assume that in E^3 some cylindrical coordinate system $(\rho, \varphi,$

$h)$, in which I is the axis h. For each φ we construct a maximal rectangle $P_0(\varphi)$ such that one of its sides (the only one attached to it) is I_0, and the mapping F is homeomorphic on that component of its preimage (we shall call this component distinguished) which has common points with I^*.

As in I, we verify that:

1°. For any $\varphi \in [0, 2\pi)$, the distinguished component of the preimage of $P_0(\varphi)$ goes to infinity.

2°. The union of the distinguished components over all $\varphi \in [0, 2\pi)$ is a domain $G_0^* \subset E^3$.

3°. The mapping F is homeomorphic in G_0^*.

4°. The union of the rectangles $P_0(\varphi)$ over all $\varphi \in [0, 2\pi)$ is a domain $G_0 \subset E^3$, contractible to I_0, and $G_0 = F(G_0^*)$.

5°. If $\rho(\varphi)$ is the length of the second side of the rectangle $P(\varphi)$, then the function $\rho(\varphi)$ is lower semicontinuous on $[0, 2\pi)$.

6°. The set $N_0 = \{\varphi \mid \varphi \in [0, 2\pi),\ \rho(\varphi) < +\infty\}$ is measurable on $[0, 2\pi)$. (As will be shown, it even has linear measure zero on $[0, 2\pi)$.)

VII. In exactly the same way as in II, we introduce the notion of a special point and verify that for any $\varphi \in [0, 2\pi)$ for which $\rho(\varphi) < +\infty$, on the side of the rectangle $P_0(\varphi)$ opposite to I_0 there is at least one special point.

VIII. Then, as in III, we show that on the distinguished component of the preimage of any sequence of points from $P_0(\varphi)$ converging to a special point there is always a sequence going to infinity.

IX. After this, relying on Lemma 2, we prove that the set

$$N_0 = \{\varphi \mid \varphi \in [0, 2\pi),\ \rho(\varphi) < +\infty\}$$

has linear measure zero on $[0, 2\pi)$.

X. Let now I_k $(k = 0, \pm 1, ...)$ be the system of intervals on I obtained from I_0 by successive shifts of I_0 along I in both directions, for example by half its length. These intervals cover the whole line I, and intervals with neighboring numbers overlap. If $P_k(\varphi)$, G_k, N_k are the symbols of the corresponding objects already constructed for I_k, then, first, evidently $G = \bigcup_k G_k$ is a domain containing I, and, second, it follows from IX that the set $N = \bigcup_k N_k$ has linear measure zero on $[0, 2\pi)$.

Thus, for almost all $\varphi \in [0, 2\pi)$,

$$P(\varphi) = \bigcup_k P_k(\varphi)$$

is a half-plane lying in G and bounded by the line I.

From Remark 2 it is easy to derive that for each φ the set $P(\varphi)$ has a component of its preimage containing I^*, on which F is homeomorphic. Consequently, the domain

$$G = \bigcup_{\varphi \in [0, 2\pi)} P(\varphi)$$

also has a component of its preimage G^*, containing I^*, on which F is homeomorphic, and moreover $F(G^*) = G$.

By construction, the domain $W = D \cup G$ is contractible to I; therefore, from the same Remark 2 it follows that F homeomorphically maps the domain $W^* = D^* \cup G^*$ onto W.

XI. It is easy to verify that the domain W is linearly connected at each of its finite boundary points, because:

XII. Each such point gives rise to, and moreover only one, attainable boundary point ([4]) of the domain W.

XIII. From XI and XII, with reference to Remark 2 and taking into account the local homeomorphy of F in E^3, we obtain that the mapping F can be extended as a homeomorphism from the domain W^* through each of its finite boundary points separately. Then it is already quite simple to verify that such an extension can be carried out simultaneously through all finite boundary points of the domain W^*. If $\widetilde{W}^*$ is the domain obtained from W^* as a result of this extension, then $F(\widetilde{W}^*) = \widetilde{W} \supset W$.

XIV. Since the complement of $\widetilde{W}$ contains no interior points, the domain $\widetilde{W}$ is obtained from W only by erasing some part of the boundary W.

Consequently, $\widetilde{W}$ also has properties XI and XII of the domain W. Hence, taking XIII into account, it follows directly that the domain $\widetilde{W}^*$ no longer admits the extension described above and, therefore, the boundary of $\widetilde{W}^*$ contains not a single finite point.

Thus, $\widetilde{W}^* \equiv E^3$, and the homeomorphy of the mapping F in the entire space is established.

XV. To complete the proof, it remains only to refer to the well-known fact that every quasiconformal homeomorphism of Euclidean space of dimension $n \geq 2$ onto itself is a corrective mapping ([2]).

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Note: Figure translations are in progress. See original paper for figures.