# ON UNITARY AND ORTHOGONAL INVARIANTS OF MATRICES

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## Abstract

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*MATHEMATICS*

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# ON UNITARY AND ORTHOGONAL INVARIANTS OF MATRICES

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**1°.** Many works have been devoted to the study of unitary and orthogonal invariants of square matrices; however, the question of finding a minimal integral rational basis and a minimal complete system of unitary and orthogonal algebraic invariants has been completely solved only for a single matrix of the second order ($^{1,2}$). For a single matrix of the third order, the corresponding invariants were considered in ($^{1,3,4}$), but in each of these works systems of invariants were obtained which contain either fewer or more invariants than a minimal complete system should contain. The set of simultaneous orthogonal invariants of a pair of $2 \times 2$ matrices given in ($^{3,5}$) is also not complete (even for real matrices).

In the present note some general properties of unitary and orthogonal invariants of square matrices are established, and not only a minimal complete system but also a minimal integral rational basis of unitary and orthogonal invariants is found for an arbitrary system of $2 \times 2$ matrices and for one matrix of type $3 \times 3$.

**2°.** By $\{A\}$ is meant a system of $n \times n$ matrices $A$ with arbitrary elements $a_{jk}$, which may take both real and complex values. Wherever the membership of the elements of the matrices in one field or another plays a role, this is stipulated separately. If $A = (a_{jk})$, then

$$A' = (a_{kj}), \qquad \bar{A} = (\bar{a}_{jk}), \qquad A^* = \bar{A}'.$$

Everywhere that, alongside the system $\{A\}$, a system $\{B\}$ is considered, it is assumed that a one-to-one correspondence $B \leftrightarrow A$ has been established between the matrices of this second system and the matrices of the system $\{A\}$. All invariants considered will be assumed to be **algebraic** (polynomial) and **absolute**. By a **polynomial in matrices** is meant a polynomial in the elements of these matrices. All polynomials are considered over an arbitrary number field $P$.

A polynomial $f(\{A\})$ is called an **affine** (**orthogonal**) **invariant** of the system $\{A\}$ if

$$f(\{V^{-1}AV\}) = f(\{A\})$$

for all values of the elements of the matrices of the system $\{A\}$ and all nonsingular (orthogonal) matrices $V$. A polynomial $f(\{A\}, \{A^*\})$ is called a **unitary invariant** of the system $\{A\}$ if

$$f(\{U^*AU\}, \{U^*A^*U\}) = f(\{A\}, \{A^*\})$$

for all complex $A$ and unitary $U$. In particular, the polynomial $f$ may depend only on $\{A\}$ or only on $\{A^*\}$.

By an **integral rational** (**functional**) **basis** of affine invariants of the system $\{A\}$ is meant such a set of affine invariants through which all the remaining affine invariants are expressed polynomially (in the form of a single-valued function). If under

if, upon removal of any invariant, an integral rational (functional) basis ceases to be such, we shall call it **minimal**. Analogous definitions are given for bases of orthogonal and unitary invariants.

Let to each matrix $A$ of the system $\{A\}$ there be put in correspondence two matrices $A^{(1)}$ and $A^{(2)}$ with numerical entries. By a **complete system** of orthogonal (unitary) invariants of the system $\{A\}$ is meant a collection of orthogonal (unitary) invariants whose equality of values for the systems $\{A^{(1)}\}$ and $\{A^{(2)}\}$ always guarantees the orthogonal (unitary) equivalence of these systems, i.e., the existence of such an orthogonal (unitary) matrix $V$ that $V^{-1}A^{(1)}V = A^{(2)}$ simultaneously for all matrices of the system $\{A\}$. If, upon removal of any invariant, a complete system ceases to be such, we shall call it **minimal**. As for affine invariants, as is known, even the set of all (algebraic) affine invariants of a matrix $A$ does not form a complete system of invariants.

3°. **Theorem 1.** *The traces of all possible elements of the multiplicative semigroup generated by the matrices of some system $\{A\}$ of variable $n \times n$ matrices $A$ form an integral rational basis of the affine invariants of the system $\{A\}$.*

The proof of this theorem is easily carried out by the same method by which it is proved for the case of one matrix in (6).

**Lemma 1.** *A polynomial $f(\{A\}, \{A^*\})$ is a unitary invariant of the system $\{A\}$ if and only if $f(\{A\}, \{B\})$ is an affine invariant of the system of matrices $\{A\} \cup \{B\}$.*

In the proof of this lemma the following two propositions play an essential role:

1. The polynomial $f(\{A\})$ is a unitary invariant of the system $\{A\}$ if and only if it is its affine invariant.

2. If $f(\{A\},\{B\})$ and $g(\{A\},\{B\})$ are two polynomials, then from the equality $f(\{A\},\{\overline{A}\}) = g(\{A\},\{\overline{A}\})$ for all complex $\{A\}$ there follows the equality $f(\{A\},\{B\}) = g(\{A\},\{B\})$ for arbitrary values of the matrices of the system $\{A\} \cup \{B\}$.

In particular, proposition 1 follows from the fact that the function

$$\varphi(V) = f(\{V^{-1}AV\}) - f(\{A\})$$

is the quotient of a polynomial in $V$ by some power of $\det V$, and therefore one may apply to it lemma (VII.1.A), p. 243 (7).

Using proposition 2 and lemma 1, it is easy to establish that the following holds.

**Theorem 2.** *The system of polynomials $\{f(\{A\},\{B\})\}$ forms an integral rational basis of the affine invariants of the system of matrices $\{A\} \cup \{B\}$ if and only if the system of polynomials $\{f(\{A\},\{A^*\})\}$ constitutes an integral rational basis of the unitary invariants of the system $\{A\}$. The minimality of either of these two integral rational bases entails the minimality of the other.*

From Theorems 1 and 2 it follows immediately that

**Corollary 1.** *The traces of all possible elements of the multiplicative semigroup generated by the matrices of the system $\{A\} \cup \{A^*\}$ form an integral rational basis of the unitary invariants of the system $\{A\}$.*

Proceeding from the general theory of invariants, we have established the analogous proposition for orthogonal invariants, i.e., have proved

**Lemma 2.** *The traces of all possible elements of the multiplicative semigroup generated by the matrices of the system $\{A\} \cup \{A'\}$ form an integral rational basis of the orthogonal invariants of the system $\{A\}$.*

From this lemma follows the corresponding result obtained in (8) for symmetric matrices, and also

**Theorem 3.** *If the system of polynomials $\{f(\{A\},\{B\})\}$ forms an integral rational basis of the affine invariants of the system of matrices $\{A\} \cup \{B\}$,*

then the system of polynomials $\{f(\{A\},\{A'\})\}$ forms an integral rational basis of the orthogonal invariants of the system $\{A\}$.

Note that even if an integral rational basis $\{f(\{A\},\{B\})\}$ is minimal, the integral rational basis $\{f(\{A\},\{A'\})\}$, generally speaking, no longer has this property.

In ($^9$) the completeness of the set of unitary invariants of a complex matrix is proved, and in ($^{10}$) the completeness of the set of orthogonal invariants of a real matrix. These results are also valid for a system of matrices. From them it follows that

**Theorem 4.** *Any integral rational basis of unitary (orthogonal) invariants of a system* $\{A\}$ *forms a complete system of unitary (orthogonal) invariants of the system* $\{A\}$ *of complex (real) matrices. The concept of a functional basis of unitary (orthogonal) invariants of a system* $\{A\}$ *of complex (real) matrices coincides with the concept of a complete system of these invariants.*

It is not difficult to show that even the set of all orthogonal invariants of a system of complex matrices is not complete.

4°. We now give the results obtained by us for an arbitrary system of $2 \times 2$-matrices $\{A_\tau\}$, in which the index $\tau$ runs through some set $T$ of finite or transfinite numbers.

**Theorem 5.** *The polynomials*

$$\operatorname{sp} A_r, \quad \operatorname{sp} A_r^2, \quad \operatorname{sp} A_r A_s \ (r < s), \quad \operatorname{sp} A_r A_s A_p \ (r < s < p)$$

$(r, s, p \in T)$ *form a minimal integral rational and minimal functional basis of affine invariants of a system of* $2 \times 2$*-matrices* $\{A_\tau\}$.

For $T = \{1, 2\}$ this basis was obtained in ($^{11}$).

From Theorems 2 and 5 it follows that

**Theorem 6.** *The polynomials*

$$\operatorname{sp} A_r, \quad \operatorname{sp} A_r^*, \quad \operatorname{sp} A_r^2, \quad \operatorname{sp} A_r^{*2}, \quad \operatorname{sp} A_r A_q^*,$$

$$\operatorname{sp} A_r A_s, \quad \operatorname{sp} A_s^* A_r^*, \quad \operatorname{sp} A_r A_s A_q^*, \quad \operatorname{sp} A_q A_s^* A_r^* \quad (r < s);$$

$$\operatorname{sp} A_r A_s A_p, \quad \operatorname{sp} A_p^* A_s^* A_r^* \quad (r < s < p)$$

$(r, s, p, q \in T)$ *form a minimal integral rational basis of the unitary invariants of a system of* $2 \times 2$*-matrices* $\{A_\tau\}$.

Hence, with the aid of additional, far from trivial arguments, the following was obtained.

**Theorem 7.** *The polynomials*

$$\operatorname{sp} A_r, \quad \operatorname{sp} A_r^2, \quad \operatorname{sp} A_r A_r^*, \quad \operatorname{sp} A_r A_s \quad (r < s); \quad \operatorname{sp} A_r A_s^* \quad (r < s);$$

$$\operatorname{sp} A_r A_q A_q^* \quad (q \neq r); \quad \operatorname{sp} A_r A_s A_p \quad (r < s < p)$$

$(r, s, p, q \in T)$ *form a minimal functional basis, and therefore also a minimal complete system of unitary invariants of a system of* $2 \times 2$*-matrices* $\{A_\tau\}$*. Here each of the* $\operatorname{sp} A_r A_s A_p$ *may be replaced by any of the three traces*

$$\operatorname{sp} A_r^* A_s A_p, \quad \operatorname{sp} A_r A_s^* A_p, \quad \operatorname{sp} A_r A_s A_p^*.$$

*Moreover, the replacement of any invariants by their conjugates is admissible.*

For $T = \{1\}$, Theorems 6 and 7 contain the results of the papers ($^1$,$^2$ ).

Taking into account Theorems 3 and 5 and investigating the syzygies arising between $\operatorname{sp} A_r A_s A_p$ and the polynomials

$$\operatorname{sp} A_r' A_s A_p, \quad \operatorname{sp} A_r A_s' A_p, \quad \operatorname{sp} A_r A_s A_p' \quad (r < s < p), \tag{1}$$

it is established that

**Theorem 8.** *The polynomials*

$$\operatorname{sp} A_r, \quad \operatorname{sp} A_r^2, \quad \operatorname{sp} A_r A_r', \quad \operatorname{sp} A_r A_s \quad (r < s); \quad \operatorname{sp} A_r A_s' \quad (r < s)$$

and $\operatorname{sp} A_r A_q A_q'$ $(q \neq r)$, together with the traces (1) $(r, s, p, q \in T)$, form a minimal integral rational basis of the orthogonal invariants of the system of $2 \times 2$ matrices $\{A_t\}$. Here each of $\operatorname{sp} A_r A_s A_p$ $(r < s < p)$ may be replaced by one of the three traces (1).

Taking into account the arguments carried out in the proof of Theorem 7, we obtain from this

**Theorem 9.** The polynomials

$$\operatorname{sp} A_r, \quad \operatorname{sp} A_r^2, \quad \operatorname{sp} A_r A_r', \quad \operatorname{sp} A_r A_s \quad (r < s); \quad \operatorname{sp} A_r A_s' \quad (r < s);$$

$$\operatorname{sp} A_r A_q A_q' \quad (q \neq r); \qquad \operatorname{sp} A_r A_s A_p \quad (r < s < p)$$

$(r, s, p, q \in T)$ form a minimal functional basis, and consequently, in the case of real matrices, a minimally complete system of orthogonal invariants of the system of $2 \times 2$ matrices $\{A_t\}$. Here each of $\operatorname{sp} A_r A_s A_p$ may be replaced by any of the three traces (1).

**Remark.** All the remaining minimal integral rational and functional bases, as well as the minimally complete systems of affine, orthogonal, and unitary invariants of a system of $2 \times 2$ matrices consisting of traces of products of the corresponding matrices, may be obtained from the systems mentioned in Theorems 5–9 by permuting the factors under the trace signs.

5°. Using Theorem 1 and the results of the paper [11], concerning the reducibility of traces of elements of the multiplicative semigroup generated by $3 \times 3$ matrices $A$ and $B$, we arrive at the conclusion that there exist only four minimal integral rational bases of affine invariants of the system of matrices $A$ and $B$ composed of the indicated traces. One of them consists of the polynomials

$$\operatorname{sp} A, \quad \operatorname{sp} B, \quad \operatorname{sp} A^2, \quad \operatorname{sp} AB, \quad \operatorname{sp} B^2,$$

$$\operatorname{sp} A^3, \quad \operatorname{sp} A^2 B, \quad \operatorname{sp} AB^2, \quad \operatorname{sp} B^3, \quad \operatorname{sp} A^2 B^2, \quad \operatorname{sp} ABA^2 B^2.$$

The other three are obtained from it by replacing the last two traces, or one of them, respectively, by $\operatorname{sp} ABAB$ and $\operatorname{sp} A^2 BAB^2$.

Using, for example, the first of these bases and Theorems 2, 4, and 3, one establishes

**Theorem 10.** For a $3 \times 3$ matrix $A$, the polynomials

$$\operatorname{sp} A, \quad \operatorname{sp} A^*, \quad \operatorname{sp} A^2, \quad \operatorname{sp} AA^*, \quad \operatorname{sp} A^{*2},$$

$$\operatorname{sp} A^3, \quad \operatorname{sp} A^2A^*, \quad \operatorname{sp} AA^{*2}, \quad \operatorname{sp} A^{*3}, \quad \operatorname{sp} A^2A^{*2}, \quad \operatorname{sp} AA^*A^2A^{*2}$$

form a minimal integral rational basis of unitary invariants. By discarding one polynomial from each pair of mutually conjugate polynomials, one obtains from it minimal complete systems, and consequently also minimal functional bases, of unitary invariants.

The polynomials

$$\operatorname{sp} A, \quad \operatorname{sp} A^2, \quad \operatorname{sp} AA', \quad \operatorname{sp} A^3, \quad \operatorname{sp} A^2A', \quad \operatorname{sp} A^2{A'}^2, \quad \operatorname{sp} AA'A^2{A'}^2 \tag{2}$$

form minimal integral rational and functional bases, and, in the case of real matrices, also a minimally complete system of orthogonal invariants.

The invariants of the system (2) were considered in [11].

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*Note: Figure translations are in progress. See original paper for figures.*

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