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Abstract

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MATHEMATICS

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ON UNIFORMLY NONCOERCIVE PROBLEMS FOR ELLIPTIC EQUATIONS

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1. Let $A(x, i\partial/\partial x)$ be a properly elliptic operator with complex-valued infinitely differentiable coefficients of order $2m$, defined on a smooth v-dimensional compact manifold Ω with boundary Γ. The problem

$$\begin{gathered} A(x, i\partial/\partial x)u = 0, \quad x \in \Omega; \\ B_j(x, i\partial/\partial x)u = \varphi_j, \quad x \in \Gamma,\ j = 1, 2, \dots, m, \end{gathered} \tag{1}$$

has been well studied in the case when the operators A and B_j satisfy a certain algebraic condition, called the condition of Z. Ya. Shapiro—Ya. B. Lopatinskii. If this condition is fulfilled, then problem (1) is said to be coercive. It is known that coercive problems are Noetherian, i.e., in the corresponding spaces ($u \in H^s(\Omega) \cap \{Au = 0\}$, $\varphi_j \in H^{s-m_j-1/2}(\Gamma)$, where m_j is the order of the operator B_j) the operator $B = B_1 \times B_2 \times \cdots \times B_m$ defines a continuous mapping whose kernel and cokernel are finite-dimensional. (For an extensive literature on this question see ([1,2]).)

In the present note a certain class of noncoercive problems is singled out, in which the Z. Ya. Shapiro—Ya. B. Lopatinskii condition is violated at every point of the boundary Γ, but nevertheless they are Noetherian in certain other spaces. These spaces will be defined below. Some special cases of problems of this type in two-dimensional domains, when methods of a complex variable can be applied to their study, were considered earlier by N. E. Tovmasyan ([4,5]).

In order to explain better the method used in these problems, in the first part of the paper we shall consider the case when the operator A has second order and Ω is a two-dimensional manifold.

2. Thus, let the operator $A(x, i\partial/\partial x)$ have second order. The problem studied is

$$A(x, i\partial/\partial x)u = 0, \quad x \in \Omega; \qquad B(x, i\partial/\partial x)u = \varphi, \quad x \in \Gamma. \tag{2}$$

Take an arbitrary point $x^0 \in \Gamma$. Introduce in a neighborhood of the point x^0 coordinates $t = (t_1, t_2)$ so that in the new coordinates the boundary Γ is described by the equation $t_1 = 0$ and the point x^0 goes over into the point $(0,0)$. The operator A, written in the new coordinates, will be denoted by $A(t, i\partial/\partial t)$. Consider the polynomial $A(t,\xi)$, corresponding to the operator $A(t, i\partial/\partial t)$, $\xi = (\xi_1, \xi_2)$.

We shall call a function $\Phi(t,\xi)$ **generalized homogeneous** if

$$\Phi(\lambda^{-1}t, \lambda\xi) = \lambda^\gamma \Phi(t,\xi), \quad \lambda > 0.$$

The number γ is called the **generalized order**. For any $N > 0$ the operators $A(t, i\partial/\partial t)$ and $B(t, i\partial/\partial t)$ can be represented in the form

$$A\left(t, i\frac{\partial}{\partial t}\right) = \sum_{k=0}^{N-1} A_k\left(t, i\frac{\partial}{\partial t}\right) + A_N\left(t, i\frac{\partial}{\partial t}\right),$$
$$B\left(t, i\frac{\partial}{\partial t}\right) = \sum_{k=0}^{N-1} B_k\left(t, i\frac{\partial}{\partial t}\right) + B_N\left(t, i\frac{\partial}{\partial t}\right), \tag{3}$$

where $A_k(t,\xi)$ and $B_k(t,\xi)$ are polynomials in t and ξ, which are generalized homogeneous functions of orders $2-k$, $n-k$, respectively; n is the order of the operator B; the coefficients of the operators A_N and B_N have zeros of order $N-2$ at the origin of the coordinates. The expansion (3) is obtained if the coefficients of the polynomials A and B are replaced by Taylor series and the terms having the same generalized order are collected.

Let

$$\widetilde{A}_k = A_k(t_1, -i\partial/\partial\xi_2, i\partial/\partial t_1, \xi_2), \qquad \widetilde{B}_k = B_k(t_1, -i\partial/\partial\xi_2, i\partial/\partial t_1, \xi_2).$$

Consider on the half-line $t_1 \geqslant 0$ the following triangular system of ordinary differential equations, depending on the parameter ξ_2, $-\infty < \xi_2 < +\infty$, $\xi_2 \neq 0$:

$$\begin{aligned} \widetilde{A}_0 E_0(t_1,\xi_2) &= 0, \\ \widetilde{A}_0 E_1(t_1,\xi_2) &= -\widetilde{A}_1 E_0(t_1,\xi_2), \\ &\vdots \\ \widetilde{A}_0 E_{N-1}(t_1,\xi_2) &= -\widetilde{A}_1 E_{N-2}(t_1,\xi_2) - \dots - \widetilde{A}_{N-1}(t_1,\xi_2). \end{aligned} \tag{4}$$

By virtue of the proper ellipticity of the operator A, the equation $\widetilde{A}_0(\lambda,\xi_2) = 0$ has, for each $\xi_2 \neq 0$, exactly one root $\lambda(\xi_2)$ for which $\operatorname{Im}\lambda(\xi_2) < 0$. Then

$$E_0(t_1,\xi_2) = C(\xi_2)\exp[-it_1\lambda(\xi_2)].$$

As $C(\xi_2)$ take any homogeneous function of order zero: $C(\xi_2) = C_1$ for $\xi_2 > 0$; $C(\xi_2) = C_2$ for $\xi_2 < 0$. From the construction of $\widetilde{A}_k$ it is easy to see that one can find solutions $E_k(t_1, \xi_2)$ of the system (4) such that the functions $E_k(t_1, \xi_2)$ will be generalized homogeneous of generalized order $-k$ and will decrease as $t_1 \to \infty$. In this case each of the functions E_k is determined up to a finite number of constants. In what follows only such solutions of the system (4) will be considered as possess the properties indicated above and, moreover, $E_0(t_1, \xi_2) \neq 0$ for $\xi_2 \neq 0$.

Note that the expression

$$\left[\widetilde{B}_0 E_k(t_1, \xi_2) + \widetilde{B}_1 E_{k-1}(t_1, \xi_2) + \cdots + \widetilde{B}_{kE} 0(t_1, \xi_2)\right]\Big|_{t_1=0} \tag{5}$$

is a homogeneous function of ξ_2 of order $n - k$. Let $p \geqslant 0$ be the smallest of the numbers $k \geqslant 0$ for which the expression (5) is different from zero for $\xi_2 > 0$. Analogously, q is introduced for $\xi_2 < 0$.

Lemma 1. The numbers p and q do not depend on the arbitrariness in the choice of the solutions of the system (4).

Definition 1. Problem (2) is called **uniformly noncoercive**, of order p, q, if the numbers p and q are finite and the same for all points x^0 of the boundary Γ. Let us note that such a name is justified only if at least one of the numbers p or q is strictly greater than zero, since for $p = q = 0$ this definition gives exactly the class of coercive problems.

Theorem 1. If problem (2) is uniformly noncoercive of order p, q, then for any function $\varphi \in H^{s-n-1/2-\max(p,q)}(\Gamma)$, satisfying a finite number of orthogonality conditions to certain infinitely differentiable functions, there exists a solution u in the space $H^s(\Omega)$. The homogeneous problem has a finite number of linearly independent solutions, and these solutions are infinitely differentiable up to the boundary.

We shall now explain the proof of this theorem, after which it will be easy to formulate a more general assertion. In the proof, singular integro-differential (s.i.-d.) operators of order not exceeding r, s on the closed curve $(\bar{\zeta})$ are used essentially.

As is known, the Dirichlet problem for a properly elliptic equation is coercive. Therefore there exists an operator R, which maps functions $\psi \in H^{s-1/2}(\Gamma)$ into $u \in H^s(\Omega)$ and is such that

$$AR\psi = 0, \qquad R\psi|_\Gamma = \psi + T\psi,$$

where T is a certain infinitely smoothing operator. Roughly speaking, the subsequent method consists in seeking the solution of problem (2) in the form $R\psi$ with unknown function ψ, and then for ψ one obtains the equation $BR\psi|_\Gamma = \varphi$.

Lemma 2. The operator $S : \psi \to BR\psi|_\Gamma$ is a s.p.-d. operator on Γ. If problem (2) is uniformly noncoercive of orders p, q, then S is a s.p.-d. operator of order not exceeding $n-p,\ n-q$ with nonzero symbol. In order to compute the symbol $\sigma(S^*)$ at the point $x^0 \in \Gamma$, when t_2 is taken as local coordinates on Γ, it suffices to choose the solution of system (4) so that $E_0(t_1, \xi_2)|_{t=0} = 1$ and $E_k(t_1, \xi_2)|_{t_1=0} = 0$ for $k > 0$. Then $\sigma(S^*)$ for positive ξ_2 is given by expression (5) with $k = p$, and $\sigma(S^*)$ for $\xi_2 < 0$ is given by expression (5) with $k = q$.

Now, relying on certain results from (3), it is easy to obtain a statement refining Theorem 1.

Definition 2. Let $H^{r,s}(\Gamma)$ be the space of functions on Γ, the norm in which is defined with the aid of some s.p.-d. operator Q of order not exceeding r, s, with nonzero symbol, by the formula

$$\|\varphi\|_{r,s} = \|Q\varphi\|_{L_2} + \|\varphi\|_{H^{\min(r,s)}\Gamma}.$$

For different choices of Q these norms are, of course, equivalent.

Theorem 2. Problem (2) is Noetherian in the spaces

$$\varphi \in H^{s-n-1/2+p,\ s-n-1/2+q}(\Gamma), \qquad u \in H^s(\Omega) \cap \{Au = 0\}.$$

Theorem 3. In the spaces indicated in Theorem 2, the index $\chi(A, B)$ of problem (2) is equal to

$$\chi(A, B) = \chi(D) + \chi(S),$$

where $\chi(D)$ is the index of the Dirichlet problem for the equation $Au = 0$, and $\chi(S)$ is the index of the s.p.-d. operator $S : \psi \to BD\psi|_\Gamma$.

We note that $\chi(S)$ is determined from the symbol $\sigma(S)$ by formula (4) from (3).

Corollary. The index of problem (2) does not change under smooth deformations of the operator B that leave the problem uniformly p, q-noncoercive.

As the simplest example, consider the problem

$$\Delta u = 0, \quad x \in \Omega; \qquad [\partial u/\partial n + i\,\partial u/\partial l + b(x)u]_\Gamma = \varphi,$$

where $\partial u/\partial n$ and $\partial u/\partial l$ are the derivatives in the normal direction and along the curve Γ, respectively. It is easy to see that this problem is not coercive, but one can verify that for $b(x) \neq 0$ it will be uniformly noncoercive of order $0, 1$. Since here $\chi(D) = 0$, the index of this problem is equal to $\chi(S)$. Using Lemma 2, it is not difficult to compute explicitly the symbol $\sigma(S)$, and, using formula (4) from (3), one can verify that

$$\chi(A,B) = \frac{1}{2\pi i}[\ln b(x)]_\Gamma.$$

3. Let us now pass to problem (1) for equations of arbitrary order on a manifold of dimension v. First of all we note that, without loss of generality, one may assume that the orders of the operators B_j are the same and equal to n, since our problem (1), for questions related to the index, is equivalent to the problem obtained if the boundary conditions $B_j u = \varphi_j$ are replaced by $C_{jB} ju = C_j \varphi_j$, where C_j is any elliptic s.p.-d. operator on Γ of order $n - m_j$. By B we shall denote the column vector composed of the operators B_j.

Let us take some coercive problem for the equation $Au = 0$, in which all the boundary operators D_j have one and the same order h. For example, as D_j one may take the operators obtained from the boundary conditions of the Dirichlet problem by equalizing the orders of differentiation, as was described above. We shall denote the column-vector of the operators D_j by D.

Proceeding as before, we represent the operators A and B in local coordinates in the form (3), where $A_k(t, i\partial/\partial t)$, $B_k(t, i\partial/\partial t)$ have generalized orders $2m-k$, $n-k$, respectively. We treat the operator D^* in an analogous way.

In equation (4), where ξ_2 must be replaced by $\xi' = (\xi_2, \dots, \xi_\nu)$. The unknowns $E_k(t_1, \xi')$ will now be vector-functions consisting of m components written as a row. We choose the function $E_0(t_1, \xi')$ so that

$$\widetilde{D}_0 E_0(t_1, \xi')\big|_{t_1=0} = I,$$

where I is the identity matrix. The remaining $E_k(t_1, \xi')$ are chosen so that they are generalized homogeneous functions of generalized order $-h-k$, and if one forms the formal series

$$E = \sum_{k=0}^{\infty} E_k$$

and expands the expression $\widetilde{D}E\big|_{t_1=0}$ as a sum of homogeneous terms

$$\widetilde{D}E\big|_{t_1=0} = \widetilde{D}_0 E_0\big|_{t_1=0} + (\widetilde{D}_0 E_1 + \widetilde{D}_1 E_0)\big|_{t_1=0} + \cdots,$$

then all terms except the first will be zero matrices. We now represent the expression $\widetilde{B}E\big|_{t_1=0}$ as a sum of homogeneous terms

$$\widetilde{B}E\big|_{t_1=0} = \widetilde{B}_0 E_0\big|_{t_1=0} + (\widetilde{B}_0 E_1 + B_1 E_0)\big|_{t_1=0} + \cdots, \tag{6}$$

Since the problem with boundary conditions D is coercive, there exists an operator R (a regularizer) which takes vector-functions $\psi \in H^{s-h-1/2}(\Gamma)$ into $u \in H^s(\Omega)$ and is such that

$$AR\psi = 0, \qquad DR\psi\big|_\Gamma = \psi + T\psi,$$

where T is a certain infinitely smoothing operator. Consider the operator $S : \psi \to BR\psi\big|_\Gamma$. There is an assertion, analogous to Lemma 2, that S is a p.i.-d. system, and if the parameters on Γ are chosen to be $t_2, \ldots, t_\nu$, then the corresponding functions K_N in the expansion (1) from (3) for the operator S^*, computed at the point $t_2 = 0, \ldots, t_\nu = 0$, are given by the sum of the first N terms of the series (6).

Definition 3. Problem (1) is called **uniformly noncoercive** if the system S is uniformly nonelliptic in the sense of (3).

Using the spaces $H^s(\Gamma)$, which were defined in (3), one can now establish the following assertion.

Theorem 4. *If problem (1) is uniformly noncoercive, then it defines a Noetherian operator from*

$$H^s(\Omega) \cap \{Au = 0\}$$

to

$$\overline{H}^{s-h-1/2}(\Gamma).$$

The kernel of this operator consists of functions infinitely differentiable up to the boundary, and the image is determined by conditions of the type of orthogonality to certain infinitely differentiable functions.

Remark. An analogous theorem is valid for arbitrary properly elliptic systems for which there exists at least one coercive problem. Moreover, all results remain valid if spaces of type H^s are replaced by spaces of type $C^{s,\alpha}$, $s > 0$.

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* Generally speaking, B and D will now be p.i.-d. operators. Therefore the possibility of representing them in the form (3) follows from the definition of a p.i.-d. operator.