A PRIORI ESTIMATES AND SOLVABILITY OF QUASILINEAR ELLIPTIC EQUATIONS OF SECOND ORDER IN A COMPOSITE DOMAIN WITH A NONLINEAR BOUNDARY CONDITION AND A CONJUGATION CONDITION

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Abstract

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A PRIORI ESTIMATES AND SOLVABILITY OF QUASILINEAR ELLIPTIC EQUATIONS OF SECOND ORDER IN A COMPOSITE DOMAIN WITH A NONLINEAR BOUNDARY CONDITION AND A CONJUGATION CONDITION

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1°. In the present work we investigate the solvability of the Dirichlet problem and the problem with oblique derivative for a quasilinear elliptic equation of second order in divergence form in a composite domain with a nonlinear conjugation condition. In the linear case such problems have been well studied (see, for example, $^{(2-6)}$).

We shall adhere to the notation of the monograph $^{(1)}$. Let T be some bounded m-dimensional domain with boundary $\mathfrak{F}T$, and let inside it there be an $(m-1)$-dimensional surface γ, dividing T into a finite number of domains:

$$T = \bigcup_{q=1}^{N} T^{(q)},$$

where the surfaces forming γ have no common points either with one another or with $\mathfrak{F}T$. Without loss of generality we shall consider the case $N = 2$. Let the functions $u^{(q)}(x)$ be given in the closed domain $\overline{T}^{(q)}$ (here and below $q = 1, 2$); suppose that $u^{(q)}(x) \in C^{(n,\lambda)}(\overline{T}^{q})$, $0 < \lambda < 1$. Denote by $U_k^{(q)}$, $U_{k,\lambda}^{(q)}$ $(0 \leq k \leq n)$ respectively the quantities $U_k(\overline{T}^{(q)})$, $U_{k,\lambda}(\overline{T}^{(q)})$, defined in § 33 of $^{(1)}$. Put

$$\mathfrak{U}_k = \mathfrak{U}_k(\overline{T}) = U_k^{(1)} + U_k^{(2)}, \qquad \mathfrak{U}_{k,\lambda} = \mathfrak{U}_{k,\lambda}(\overline{T}) = U_{k,\lambda}^{(1)} + U_{k,\lambda}^{(2)}, \qquad k \leq n. \quad (1)$$

Consider the space—the direct sum:

$$C^{(n,\lambda)}(\overline{T}) = C^{(n,\lambda)}(\overline{T}^{(1)}) + C^{(n,\lambda)}(\overline{T}^{(2)}).$$

Define the norm in $C^{(n,\lambda)}(\overline{T})$ by the formula

$$\|u\| = \sum_{k=0}^{n} \mathfrak{U}_k + \mathfrak{U}_{n,\lambda}.$$

The space $C^{(n,\lambda)}(\overline{T})$ will be Banach.

The aim of our work is to prove an existence and uniqueness theorem "in the large" for a classical solution of the following boundary-value problem:

$$\begin{gathered}\sum_{i=1}^{m} \frac{d}{dx_i} a_i^{(q)}(x, u^{(q)}, u_x^{(q)}) + a^{(q)}(x, u^{(q)}, u_x^{(q)}) = 0, \qquad x \in T^{(q)};\\ [u]_\gamma = 0; \qquad \left[p(x,u) \sum_{i=1}^{m} a_i(x,u,u_x)\gamma_i(x) + \psi(x,u) \right]_\gamma = 0; \qquad \text{(I)}\\ \sum_{i=1}^{m} a_i^{(2)}(x, u^{(2)}, u_x^{(2)})\gamma_i(x) + \varphi(x, u^{(2)}) \Big|_{\mathfrak{F}T} = 0,\end{gathered}$$

where

$$x = (x_1, \dots, x_m); \qquad u_x = (u_{x_1}, \dots, u_{x_m});$$

$\gamma_i(x)$ $(i = 1, \dots, m)$ are the components of the normal, exterior with respect to $T^{(1)}$ $(T^{(2)})$, to the surface γ $(\mathfrak{F}T)$ at the point x;

$$[v]_\gamma = v^{(1)}|_\gamma - v^{(2)}|_\gamma,$$

where

$$v^{(q)}|_\gamma = \lim_{T^{(q)} \ni x \to y \in \gamma} v^{(q)}(x) \qquad (q = 1, 2).$$

2°. In this section we formulate exact a priori Schauder estimates for the solution of the linear boundary-value problem in a composite domain:

$$\begin{gathered}\sum_{i=1}^{m} a_{ik}^{(q)}(x) u_{x_i x_k}^{(q)} + \sum_{k=1}^{m} b_k^{(q)}(x) u_{x_k}^{(q)} + c^{(q)}(x) u^{(q)} = f^{(q)}(x), \qquad x \in T^{(q)};\\ [u]_\gamma = \chi(x); \qquad \left[\sum_{k=1}^{m} \tau_k(x) u_{x_k} + \sigma(x) u(x) \right]_\gamma = \psi(x), \qquad x \in \gamma; \qquad \text{(II)}\\ \sum_{k=1}^{m} v_k(x) u_{x_k}^{(2)} + \mu(x) u^{(2)} \Big|_{\mathfrak{F}T} = \varphi(x), \qquad x \in \mathfrak{F}T.\end{gathered}$$

Assumptions:

1) the ellipticity condition in the form

$$\sum_{i,k=1}^{m} a_{ik}^{(q)}(x) \xi_i \xi_j \geq \nu^{(q)} \sum_{i=1}^{m} \xi^2, \qquad \nu^{(q)} = \text{const} > 0 \quad (q = 1, 2);$$

2)

$$a_{ik}(x),\ b_k(x),\ c(x),\ f(x) \in C^{(0,\lambda)}(\overline{T});$$

$$\tau_k(x),\ \sigma(x) \in C^{(1,\lambda)}(\overline{T}); \qquad \chi(x) \in C^{(2,\lambda)}(\gamma);$$

$$\psi(x) \in C^{(1,\lambda)}(\gamma); \qquad v_k(x),\ \mu(x),\ \varphi(x) \in C^{(1,\lambda)}(\mathfrak{F}T)\ (i,k = 1,\dots,m)$$

3)

$$\sum_{i=1}^{m} \tau_i^{(q)}(x)\gamma_i(x) \geq \nu_0^{(q)} = \text{const} > 0, \qquad x \in \gamma \quad (q = 1,2);$$

$$\sum_{i=1}^{m} v_i(x)\gamma_i(x) \geq \nu_0 = \text{const} > 0, \qquad x \in \mathfrak{F}T.$$

4) $\mathcal{A}_0 + \mathcal{T}_0 + \mathcal{N}_0 = O(1)$, where $\mathcal{A}_0, \mathcal{T}_0, \mathcal{N}_0$ are quantities referring respectively to $a_{ik}(x), \tau_k(x), v_k(x)$ and defined with account taken of (1).

5) $T^{(q)}$ are domains of class $A^{(2,\lambda)}$ (see (1)).

Theorem 1. Under assumptions 1)–5), for every possible solution of problem (II) of class $C^{(2,\lambda)}(\overline{T})$ the estimate

$$\begin{gathered}
\mathfrak{U}_{2,\lambda}(\overline{T}) = O\{(\mathcal{A}_{0,\lambda} + \mathcal{T}_{0,\lambda} + \mathcal{N}_{0,\lambda} + 1)\mathfrak{F}_0 + \mathfrak{F}_{0,\lambda} + \mathfrak{X}_{2,\lambda} + \\
+(\mathcal{A}_{0,\lambda} + \mathcal{T}_{0,\lambda} + 1)\Psi_1 + \Psi_{1,\lambda} + (1 + \mathcal{N}_{1,\lambda})\Phi_0 + \\
+(\mathcal{A}_{0,\lambda} + \mathcal{N}_{0,\lambda} + 1)\Phi_1 + \Phi_{1,\lambda} + \mathfrak{U}_0 \cdot [(\mathcal{A}_{0,\lambda} + \mathcal{T}_{0,\lambda} + \mathcal{N}_{0,\lambda} + 1)\mathcal{C}_0 + \\
+\mathcal{C}_{0,\lambda} + (\mathcal{A}_{0,\lambda} + \mathcal{T}_{0,\lambda} + 1)\mathcal{S}_1 + \mathcal{S}_{1,\lambda} + (1 + \mathcal{N}_{1,\lambda})\mathcal{M}_0 + \\
+(\mathcal{A}_{0,\lambda} + \mathcal{N}_{0,\lambda} + 1)\mathcal{M}_1 + \mathcal{M}_{1,\lambda} + (\mathcal{C}_0 + \mathcal{S}_1 + \mathcal{M}_1)^{(2+\lambda)/2} + 1] \\
+\mathfrak{U}_1 \cdot [(\mathcal{A}_{0,\lambda} + \mathcal{T}_{0,\lambda} + \mathcal{N}_{0,\lambda} + 1)\mathcal{B}_0 + \mathcal{B}_{0,\lambda} + \mathcal{S}_{0,\lambda} + \mathcal{M}_{0,\lambda} + \mathcal{T}_{1,\lambda} + \\
+\mathcal{N}_{1,\lambda} + (\mathcal{A}_{0,\lambda} + \mathcal{T}_{0,\lambda} + 1)(\mathcal{S}_0 + \mathcal{T}_1 + 1) + (\mathcal{A}_{0,\lambda} + \mathcal{N}_{0,\lambda} + 1)(\mathcal{M}_0 + 1) + \\
+(\mathcal{B}_0 + \mathcal{S}_0 + \mathcal{M}_0 + \mathcal{T}_1 + \mathcal{N}_1 + 1)^{1+\lambda} + \\
+(\mathcal{A}_{0,\lambda} + \mathcal{T}_{0,\lambda} + \mathcal{N}_{0,\lambda} + 1)^{(1+\lambda)/\lambda}\}, \qquad (2)
\end{gathered}$$

holds, depending on m, the domain T, and the constants entering the assumptions of item 2°; the quantities on the right in (2) have the same meaning as in § 35 of (1), with account taken of (1).

From estimate (2), by a known method (1), one can obtain the usual Schauder estimate. We note that, in the case of a single domain, estimates of type (2) were established in § 35 of (1) for the Dirichlet problem and in (7) for the third boundary-value problem (see also (3.5) in (2)). Estimate (2) is established analogously to (1,7), using the results of (3).

The explicit dependence on the coefficients of problem (II) in estimate (2)—in this sense we call estimate (2) exact—makes it possible to prove a conditional existence theorem for a general quasilinear problem in a composite domain. For lack of space we omit the formulation of this conditional theorem. We note only that it is a theorem of the type of Theorem 42, IV in (^{1}) and Theorem 2 in (^{7}) (see also Theorem 1.2, Chap. X in (^{2})), concerning the first and third boundary-value problems, respectively, in a single domain.

3°. In essence, the conditional theorem asserts that the general quasilinear problem will be solvable if there are a priori estimates for the quantities $\mathfrak{U}_0, \mathfrak{U}_1, \mathfrak{U}_{1,\lambda}$. Such estimates can be established in a special case of the general quasilinear problem, namely for problem (I), when the equations are in divergence form and the conormal derivative appears in the boundary condition and in the conjugation condition.

First of all, relying on the maximum principle for elliptic equations, one can establish an estimate for the quantity $\mathfrak{U}_0$.

Having obtained an estimate for $\mathfrak{U}_0$, we, following De Giorgi' s idea (see (^{2})), obtain an estimate for the quantity $\mathfrak{U}_1$, and then also an estimate for the quantity $\mathfrak{U}_{1,\lambda}$. The required estimates are established by us in a neighborhood of the surface γ separating the domains $T^{(1)}, T^{(2)}$, since estimates inside $T^{(q)}$ and near the surface $\mathfrak{F}T$ were established in (^{2}).

Let

$$\mathfrak{M}_0^{(q)} = \{x \in \bar{T},\ \mathfrak{U}_0 \leq M_0;\ |\nabla u^{(q)}| < \infty\}.$$

Assumptions:

1)

$$a_i^{(q)},\ a^{(q)} \in C^{(1)}(\bar{T}^{(q)} \times R^{m+1}); \quad p^{(q)}(x, u^{(q)}),\ \psi^{(q)}(x, u^{(q)}) \in C^{(2)}(\bar{T}^{(q)} \times R),$$

$$\varphi(x, u^{(2)}) \in C^{(2)}(\bar{T}^{(2)} \times R);$$

2) on the set $\mathfrak{M}_0^{(q)}$ the inequalities

$$\sum_{i,j=1}^{m} \frac{\partial a_i^{(q)}(x, u^{(q)}, u_x^{(q)})}{\partial u_{x_j}^{(q)}} \xi_i \xi_j \geq \delta_0^{(q)}(|u^{(q)}|)(1 + |\nabla u^{(q)}|)^{k-2} \cdot \sum_{i=1}^{m} \xi_i^2;$$

$$(1 + |\nabla u^{(q)}|)^2 \sum_{i,j=1}^{m} \left| \frac{\partial a_i^{(q)}}{\partial u_{x_j}^{(q)}} \right| + (1 + |\nabla u^{(q)}|) \sum_{i,j=1}^{m} \left[|a_i^{(q)}| + \left| \frac{\partial a_i^{(q)}}{\partial x_j} \right| + \right.$$

$$\left. + \left| \frac{\partial a_i^{(q)}}{\partial u^{(q)}} \right| + \left| \frac{\partial a^{(q)}}{\partial u_{x_j}^{(q)}} \right| \right] + |a^{(q)}| + \sum_{j=1}^{m} \left| \frac{\partial a^{(q)}}{\partial x_j} \right| + \left| \frac{\partial a^{(q)}}{\partial u^{(q)}} \right| \leq$$

$$\leq \delta^{(q)}(|u^{(q)}|)(1 + |\nabla u^{(q)}|)^k, \qquad k > 1 \quad (q = 1, 2),$$

where $\delta_0^{(q)}(t)$, $\delta^{(q)}(t)$ are positive, respectively nonincreasing and nondecreasing continuous functions defined for $t \geq 0$;

3)
$$p^{(q)}(x, u^{(q)}) \geq p_0(M_0) = \text{const} > 0; \qquad x \in \bar{T}^{(q)};$$

4)
$$\max_{\substack{x\in\bar{T};\ \mathfrak{U}_0 \leq M_0 \\ i,j=1,\dots,m}} \left\{ p^{(q)}(x, u^{(q)}),\ \left|\frac{\partial p^{(q)}}{\partial x_i}\right|,\ \left|\frac{\partial p^{(q)}}{\partial u^{(q)}}\right|,\ \left|\frac{\partial^2 p^{(q)}}{\partial x_i \partial x_j}\right|,\right.$$
$$\left.\left|\frac{\partial^2 p^{(q)}}{\partial x_i \partial u^{(q)}}\right|,\ \left|\frac{\partial^2 p^{(q)}}{\partial u^{(q)2}}\right|\right\} \leq \delta^{(q)}(M_0) = \text{const} > 0;$$

analogous inequalities for $\psi^{(q)}(x, u^{(q)})$ and $\varphi(x, u^{(2)})$;

5) $T^{(q)} \in A^{(2)}$.

Theorem 2. Suppose the assumptions of item 3° are satisfied. Let $u(x) \in C^{(2)}(\bar{T})$ be a possible solution of problem (I), and let $\mathfrak{U}_0 \leq M_0$. Then the quantities $\mathfrak{U}_1, \mathfrak{U}_{1,\lambda}$ are estimated, for some $\lambda > 0$, by a constant M_1 depending only on $M_0, k, \delta_0(M_0) = \min(\delta_0^{(1)}, \delta_0^{(2)})$; $\delta(M_0) = \max(\delta^{(1)}, \delta^{(2)})$, and the domain T. These same quantities also determine the exponent $\lambda > 0$.

Finally, on the basis of Theorem 2, from the conditional theorem there follows the main

Theorem 3. Suppose the assumptions of Theorem 2 are satisfied. Let λ, M_0, M_1 be the numbers determined by this theorem. Form the set

$$\mathfrak{M} = \{x \in \bar{T}; \quad \mathfrak{U}_0 \leq M_0; \quad \mathfrak{U}_1 \leq M_1\}$$

and suppose, in addition, that the conditions are fulfilled:

1) on $\mathfrak{M}$ there exist all possible first- and second-order partial derivatives of the functions $a_i^{(q)}, p^{(q)}, \psi^{(q)}, \varphi$, and the first derivatives of $a_i^{(q)}$ with respect to $u^{(q)}, u_{x_k}^{(q)}$, which on $\mathfrak{M}$: a) are bounded in absolute value; b) satisfy the Hölder condition with exponent λ in x_i $(i = 1, \dots, m)$; c) satisfy the Lipschitz condition in the arguments $u^{(q)}, u_{x_i}^{(q)}$;

2)
$$\frac{\partial}{\partial u^{(2)}}(p^{(2)}(x, u^{(2)})\varphi(x, u^{(2)})) \geq 0, \quad x \in \mathfrak{F}T, \quad \mathfrak{W}_0 \leq M_0;$$

3)
$$-\partial a^{(q)}(x, u^{(q)}, u_x^{(q)})/\partial u^{(q)} \geq c^{(q)} > 0$$

on $\mathfrak{M}$, where $c^{(q)} = \text{const} > 0$ and depend on M_0, M_1 and on the constants from the assumptions of item 3°, whose explicit form we omit for brevity of exposition.

4) $T^{(q)} \in A^{(2,\lambda)}$ $(q = 1, 2)$.

Then there exists a unique solution $u(x) \in C^{(2,\lambda)}(\overline{T})$ of problem (I).

The Dirichlet problem with the boundary condition $u^{(2)}|_T = \varphi(x)$ is studied analogously. In this case one uses the estimates of § 35 in (^{1}) and Chapter IV of (^{2}).

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