SYMMETRIZABLE* SPACES AND ε-MAPS

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MATHEMATICS

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SYMMETRIZABLE* SPACES AND ε-MAPS

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Let X be a symmetrizable space with symmetric d and $\varepsilon > 0$.

Definition 1. A map f of the space X onto a topological space Y will be called a **weak ε-map** if for every $y \in Y$ there is a point $x \in X$ such that

$$f^{-1}y \subseteq O_\varepsilon x = E\{z: \ z \in X, \ d(x, z) < \varepsilon\}.$$

Definition 2. A map f of the space X onto Y is called an **ε-map** if for every $y \in Y$ $\operatorname{diam}(f^{-1}y) \le \varepsilon$.

Theorem. *If X is a Hausdorff symmetrizable space and for every n the map $f_n : X \to Y_n$ is a continuous weak $1/n$-map, then the natural map $f : X \to \prod_{n=1}^{\infty} Y_n$ is an embedding; if for every n f_n is a continuous closed $1/n$-map, then f is a homeomorphism.*

Proof. It is necessary to show that f is one-to-one. Suppose that $x \in X$, $y \in X$ and $fx = fy$, i.e. $f_{nx} = f_{ny}$ for every n. Since f_n is a weak $1/n$-map, for each n there exists a point x_n such that $d(x_n, x) = 1/n$, $d(x_n, y) = 1/n$, i.e. $x = \lim x_n$, $y = \lim_{n\to\infty} x_n$, which is possible only when $x = y$, since, by assumption, X is Hausdorff. Thus, f is an embedding.

Suppose now that f_n is a closed $1/n$-map for each n. Let F be a closed subset of X. Put $\Phi = fX \cap \left(\prod_{n=1}^{\infty} f_{nF}\right)$ and show that $fF = \Phi$. It is easy to see that $fF \subseteq \Phi$, and therefore let $y \in \Phi$. Consequently, $y = fx = \{f_{nx}\}$ for some $x \in X$, for which $f_n^{-1}(f_{nx}) \cap F \neq \Lambda$. Let $z_n \in f_n^{-1}(f_{nx}) \cap F$. Since f_n is a $1/n$-map, $d(x, z_n) \le 1/n$. Consequently, $x \in F$ ($x = \lim_{n\to\infty} z_n$, $z_n \in F$). Thus, fF is closed in fX, since Φ is closed in fX, i.e. f is a homeomorphism of X onto fX.

In what follows, let P denote such a property of topological spaces that: a) if X is a space with property P, then every subspace X_0 is also a space with property P, and b) if for every n X_n is a space with property P, then $\prod_{n=1}^{\infty} X_n$ is also a space with property P.

Corollary 1. *If X is a Hausdorff symmetrizable space, then X embeds in a space with property P if and only if for every $\varepsilon > 0$ it admits a weak ε-map onto a space with property P.*

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* We consider spaces symmetrizable in the sense of A. V. Arhangel' skii (see (2)): a space X is called symmetrizable with symmetric d if: 1) $d(x,y) \geq 0$, $d(x,y) = d(y,x)$ for any $x \in X$, $y \in X$; 2) $d(x,y) = 0$ if and only if $x = y$; 3) if U is an open subset of X and $x \in U$, then there exists $\varepsilon > 0$ such that $O_\varepsilon x \subseteq U$; 4) if $A \subset X$, A is not closed and is nonempty, then there exists $x \in X \setminus A$ such that $d(x, A) = 0$.

In particular, X is compactified onto a metrizable space if and only if, for every $\varepsilon > 0$, it admits a weak ε-mapping onto a metrizable space. Moreover: X has property P if and only if, for every $\varepsilon > 0$, it admits a closed ε-mapping onto a space with property P.

In particular, X is metrizable if and only if, for every $\varepsilon > 0$, it admits a closed ε-mapping onto a metrizable space.

Corollary 2. (J. Ceder). *A symmetrizable paracompact space with the first axiom of countability (a strongly symmetrizable paracompact space) admits a compactification onto a metrizable space.*

The last assertion can be strengthened as follows: if X is a symmetrizable paracompact space with the first axiom of countability and, for each $n = 1, 2, ...$, ω_n denotes an open cover of the space X, then there exists a metrizable space R and a compactification $f : X$ onto R such that, for each n, f is an ω_n-mapping.

Proof. Put $\eta_n = \{\langle O_{1/n} x \rangle,\ x \in X\}$. Next we construct a sequence of open covers $\gamma_1, \gamma_2, ...$ such that:
1) γ_1 is star-refined into $\omega_1 \wedge \eta_1$;
2) γ_{k+1} is star-refined into $\gamma_k \wedge \omega_{k+1} \wedge \eta_{k+1}$.

On the set X we introduce a symmetric function $D(x,y)$ as follows:

1°. If for two given points x and y there exists no element of the cover γ_1 containing both these points, set $D(x,y) = D(y,x) = 1$.

2°. Suppose case 1° does not occur; then there exists $k \geq 1$ such that both points x and y are contained in some element of the cover γ_k. We show that if $x \neq y$, then among these k there is a largest $k = k(x,y) = k(y,x)$. Indeed, otherwise for every k there would exist points x_k such that $x \in O_{1/k} x_k$, $y \in O_{1/k} x_k$, which is impossible, since X is a Hausdorff space. Put then $D(x,y) = 1/2k(x,y)$ and $D(z,z) = 0$ for every $z \in X$. The symmetry of the function D is obvious; it is also clear that $D(x,y) = 0$ if and only if $x = y$. Consequently, we may consider the space $X; D$, whose topology is defined as follows: a subset $A \subseteq X$ is called closed if and only if from $x \in X$ and $x \notin A$ it follows that $D(x, A) > 0$.

The following property of the function $D(x,y)$ is easily proved: if $D(x,y) \leq$

$1/2^{k+1}$, $D(y,z) \leq 1/2^{k+1}$, then $D(x,z) \leq 1/2^k$. Hence it follows that the function D satisfies the following condition.

Limit axiom. If $\lim_{n\to\infty} D(x,x_n) = 0$ and $\lim_{n\to\infty} D(x_n,y_n) = 0$, then $\lim_{n\to\infty} D(x,y_n) = 0$.

Applying a theorem of A. V. Arhangel' skii ([3]), we obtain that the space $(X;D)$ is strongly symmetrizable with the symmetric D and is metrizable. The identity mapping f of the space X onto the space $(X;D)$ is a compactification, since the relation

$$E\{z:\ z \in X,\ D(x,z) < 1/2^k\} = \mathrm{St}_{\gamma_{k+1}}\, x \tag{1}$$

holds for every point x and for every k, by construction.

It remains to show that the mapping f is an ω_n-mapping for each n. But this follows immediately from the fact that relation (1) holds and that the cover γ_{k+1} is star-refined into ω_{k+1}.

Thus Corollary 2 is proved. If it is applied in the special case when X is bicompact, we obtain the theorem of B. Nemitskii ([5]): *a symmetrizable bicompact space with the first axiom of countability is metrizable.* This is so because a compactification of a bicompact space is a homeomorphism. Similarly, if we take into account a theorem of A. V. Arhangel' skii ([1]), asserting that a feathered paracompact space which is compactified onto a metrizable space—

···property, is metrizable, we obtain A. V. Arhangel' skii' s theorem ([2]): *a symmetrizable feathered paracompact space with the first axiom of countability is metrizable.*

In connection with the preceding, the following proposition is of interest.

Proposition. *If X is a symmetrizable space whose symmetrix d is such that for every point $x \in X$ and for every $\varepsilon > 0$ there is a $\delta = \delta(x,\varepsilon) > 0$ such that from $d(x,y) < \delta$ it follows that $d(x,y) < d(y, X \setminus O_\varepsilon x)$, then X is paracompact with the first axiom of countability.*

Proof. We first show that the symmetrix d agrees with the topology of the space X in the strong sense. For this we prove the following lemma:

Lemma. *If a space Z is symmetrizable with symmetrix ρ, then Z is symmetrizable in the strong sense with symmetrix ρ if and only if ρ satisfies the following condition*

$$(\alpha) \quad \text{If } \lim_{n\to\infty} \rho(x,x_n) = 0 \text{ and for every } n \ \lim_{k\to\infty} \rho(x_n, x_{n,k}) = 0,$$
$$\text{then there exist sequences of natural numbers } \{m_n\}_{n=1}^{\infty},\ \{k_n\}_{n=1}^{\infty}$$
$$\text{such that } \lim_{n\to\infty} \rho(x, x_{m_n,k_n}) = 0.$$

Indeed, suppose that ρ satisfies condition (α) and let $A \subset Z$. Put $[A]^1 = E\{x : \rho(x, A) = 0\}$. Since ρ satisfies condition (α), we have $[[A]^1]^1 = E\{x : \rho(x, [A]^1) = 0\} = [A]^1$, and hence $[A]^1$ is closed, i.e. ρ agrees with the topology of Z in the strong sense ($x \in [A]$ if and only if $\rho(x, A) = 0$).

If Z is symmetrizable in the strong sense with symmetrix ρ and $\lim_{n\to\infty} \rho(x, x_n) = 0$, $\lim_{k\to\infty} \rho(x_m, x_{m,k}) = 0$, then there exist m_n and k_n such that, for $m \geq m_n$ and $k > k_n$, $x_{m,k} \in (O_{1/n}x)$. Hence $\lim_{n\to\infty} \rho(x, x_{m_n,k_n}) = 0$, i.e. ρ satisfies condition (α).

It is easy to verify that the symmetrix d under the assumptions of the proposition satisfies condition (α).

We now show that the space X is collectively normal. From this, by the Mack-Olin-Bing theorem (see ([5])), our assertion will follow. Let $\{F_\alpha\}$ be a discrete family of closed subsets of the space X.

Put

$$U_\alpha = E\{x : \ d(x, F_\alpha) < d(x, \bigcup_{\beta\neq\alpha} F_\beta)\}$$

and show that

$$V_\alpha = \langle U_\alpha \rangle \supset F_\alpha.$$

This assertion will prove the claim, since the V_α are pairwise disjoint (because the U_α, obviously, are pairwise disjoint). Let $x \in F_\alpha$, and choose $\varepsilon(x) > 0$ so that

$$O_{\varepsilon(x)}x \cap \left(\bigcup_{\beta\neq\alpha} F_\beta \right) = \Lambda.$$

We have

$$V_\alpha \supset \bigcup_{x\in F_\alpha} \langle O_{\delta(x,\varepsilon(x))}x \rangle \supset F_\alpha,$$

where $\delta(x, \varepsilon(x))$ is chosen according to the condition of the proposition, as was required to prove.

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* Therefore, if d is also a Cauchy symmetrix (see ([7])), i.e. if every convergent sequence is fundamental, then X will be metrizable.

Note: Figure translations are in progress. See original paper for figures.