# A CONTACT ELASTIC-PLASTIC PROBLEM FOR PLATES

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Fig. 1

Figure 1: Fig. 1

## Abstract

## Full Text

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# A CONTACT ELASTIC-PLASTIC PROBLEM FOR PLATES

In the present note an exact solution is obtained for the contact problem for bodies made of an ideal elastic-plastic material obeying the Tresca–Saint-Venant plasticity condition. It is assumed that the deformable bodies are in a state of plane stress and that there is no friction on the contact area.

§ 1. Let two straight cylinders of equal length touch coaxially by their lateral surfaces, so that the contact area is a part of the cylindrical surface bounded by the ends of the cylinders and by some of their generators (Fig. 1). The ends of the cylinders are assumed to be free of loads, and the size of the contact area transverse to the generators, as usual ([1]), is considered small in comparison with the characteristic radius of curvature of the lateral surface. Under these conditions the deformable bodies in the neighborhood of the contact area will be in a plane-stress state, and the boundary conditions in this neighborhood may be transferred to the plane $y = 0$. The material of the bodies is assumed to obey the Tresca–Saint-Venant plasticity condition, according to which the maximum shear stress at every point of the body does not exceed a certain constant value for each material (the yield limit). Friction on the contact area is neglected. In a rectangular Cartesian coordinate system $xy$, the deformable bodies may be regarded as half-planes $y > 0$ and $y < 0$, respectively. We shall agree to mark all quantities referring to the lower and upper half-planes with the indices 1 and 2, respectively.

Fig. 1

We shall show that, in the problem posed, the plastic regions are lines of zero thickness situated on the $x$-axis, whose length is determined in the course of the solution. For this we use the following general proposition ([2]). Let an elastic half-plane be subjected to the action of normal compressive loads $\sigma_y = p(x)$ ($p(x) \leq 0$) and zero tangential loads $\tau_{xy} = 0$, while the stresses at infinity

Fig. 2

Figure 2: Fig. 2

vanish; then the principal stresses $\sigma_1$ and $\sigma_2$ at each point of the half-plane will satisfy the conditions $0 \leq \sigma_{1,2} \leq p_{\max}$, where $p_{\max}$ is the largest value of the boundary load $p(x)$, and equality can be attained only on the boundary of the half-plane. From this proposition, and also from the plasticity condition, which obviously in the present case may be written in the form $|\sigma_{1,2}| \leq \sigma_s$, where $\sigma_s$ is the yield limit in compression, it follows that a solution with a plastic line on the boundary of the half-plane such that everywhere on the boundary $p(x) \leq \sigma_s$ is the exact solution of the corresponding elastic-plastic problem.

In the contact problem under consideration, the plastic line will always be realized on the contact area in the more plastic body (i.e., the body with the smaller $\sigma_s$) near the greatest concentration of the contact pressure $p(x)$. At the ends of the cylinder, at the location of the plastic line, characteristic elevations of the material will remain as a result of its being squeezed out owing to plastic flow. The indicated exact solution, ne-

continuous in the stresses and discontinuous in the normal displacement $v$, is unique for a given loading history and a given distribution of initial residual stresses, by virtue of the theorem on the uniqueness of the distribution of the rates of change of stresses in an elastic-plastic body ([3]). We shall restrict ourselves to the case of a simple loading path and zero initial stresses. It should be noted that a discontinuity in the normal component of the velocity or displacement ($v$) in the theory of the plane stress state of an ideal elastic-plastic material is admissible, since in fact it corresponds to a discontinuity in the tangential component of the velocity on the slip plane, situated at each point of the plastic region at an angle of 45° to the plane $xy$.

**Fig. 2**

From the uniqueness of the exact solution there also follows the nonexistence of a solution of the stated elastic-plastic problem that is continuous in velocities (and displacements). The physical meaning of the latter result becomes entirely obvious if one considers the equilibrium and deformation of an infinitely small element of the body near the plastic line (Fig. 2). When the plastic state is reached in this element, material is extruded in the direction normal to the plane $xy$ as a result of plastic flow. Therefore, if the rate of change of the external loads is small compared with the characteristic rate of flow (not determined within the framework of the theory of an ideally plastic body), then the plastic region can never spread over a finite area. In real materials the plastic line will have a finite thickness owing to hardening.

We note that these same considerations were used earlier to obtain the exact solution of the elastic-plastic problem for a plate with a crack ([2]).

§ 2. The stresses $\sigma_x, \sigma_y, \tau_{xy}$ and displacements $u, v$ in the problem under consideration are written in terms of Muskhelishvili potentials $\Phi_1(z)$ and $\Phi_2(z)$ as follows ([4]):

$$\sigma_x + \sigma_y = 2[\Phi(z) + \overline{\Phi}(z)],$$

$$\sigma_y - i\tau_{xy} = \Phi(z) - \Phi(\overline{z}) + (z - \overline{z})\overline{\Phi'}(z),$$

$$2\mu\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \chi\Phi(z) + \Phi(\overline{z}) - (z - \overline{z})\overline{\Phi'}(z), \tag{1}$$

$$z = x + iy, \qquad \chi = \frac{3 - \nu}{1 + \nu}.$$

Here $\mu$ is the shear modulus, and $\nu$ is Poisson' s ratio. The indices 1 and 2 have been omitted for brevity. The functions $\Phi_1(z)$ and $\Phi_2(z)$ are analytic in the entire $z$-plane, except for a cut along the $x$-axis, where the contact segment is located, and satisfy the conditions ([4])

$$\overline{\Phi_1}(z) = -\Phi_1(z), \qquad \overline{\Phi_2}(z) = -\Phi_2(z), \qquad \Phi_2(z) = -\Phi_1(z). \tag{2}$$

Using (1) and (2), the original boundary conditions:

$$\text{for } y = 0, \quad \sigma_y = \sigma_s \text{ on the plastic line } L; \tag{3}$$

$$\partial v_1/\partial x - \partial v_2/\partial x = f'(x) \quad \text{for } y = 0 \text{ on the remaining contact segment } M;$$

$$f(x) = f_2(x) - f_1(x),$$

where $y = f_1(x)$ and $y = f_2(x)$ are the equations of the lateral surfaces of the cylinders before deformation, can be written in the following form:

$$\Phi_1^- - \Phi_1^+ = \sigma_s \quad \text{on } L,$$

$$\Phi_1^+ + \Phi_1^- = \lambda i f'(x) \quad \text{on } M, \tag{4}$$

where

$$\lambda = \frac{4}{(\chi_1 + 1)/\mu_1 + (\chi_2 + 1)/\mu_2}.$$

The boundary-value problem (4) is easily solved by quadratures for any number of segments $L$ and $M$ along the real axis (by means of the device used, for example, in ([5])). The solution of this problem is sought in the class of functions bounded everywhere; the linear dimensions of the plastic lines, as well as of the contact area in the case of smooth surfaces, are determined from solvability conditions and conditions at infinity. The position and number of plastic lines are readily found from physical considerations.

Fig. 3
**Fig. 3**

Fig. 4
**Fig. 4**

We give the final formulas for two special cases of greatest interest.

1°. **Rectangular rigid punch** (Fig. 3).

$$\Phi_1(z) = \frac{\sigma_s}{2\pi i} \ln \frac{(z+l)\left[\sqrt{(l^2-a^2)(z^2-a^2)} + lz - a^2\right]}{(z-l)\left[\sqrt{(l^2-a^2)(z^2-a^2)} + lz + a^2\right]}. \quad (5)$$

Here, as $z \to \infty$, $\sqrt{z^2 - a^2} = z + O(z^{-1})$.

The size of the plastic region is determined by the formula (Fig. 3)

$$a = l\sqrt{1 - P^2/4l^2\sigma_s^2}. \quad (6)$$

2°. **Two circular cylinders of radii $R_1$ and $R_2$** (Fig. 4).

$$\Phi_1(z) = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\lambda i z - \frac{\sigma_s}{2\pi i}\sqrt{(z^2-a^2)(z^2-l^2)} \int_{-a}^{+a} \frac{dt}{\sqrt{(l^2-t^2)(a^2-t^2)}(t-z)}. \quad (7)$$

Here, as $z \to \infty$, the root before the integral behaves as $z^2$, while the root under the integral is taken to be positive. The plastic zones are shown by bold lines in Figs. 3 and 4.

The parameters $a$ and $l$, which determine the size of the contact area and of the plastic zone, are found from the system of equations

$$\frac{\pi\lambda}{\sigma_s} l \left(\frac{1}{R_1} + \frac{1}{R_2}\right) = 2K\left(\frac{a}{l}\right),$$

$$\left(\frac{a^2}{l^2} - 1\right) K\left(\frac{a}{l}\right) + 2E\left(\frac{a}{l}\right) = \frac{P}{l\sigma_s}. \tag{8}$$

Here $K$ and $E$ are complete elliptic integrals. The corresponding elastic problem was solved earlier by Föppl ([6]).

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*Note: Figure translations are in progress. See original paper for figures.*