ON BOUNDARY-VALUE PROBLEMS OF THE CONTINUUM THEORY OF DISLOCATIONS

THEORY OF ELASTICITY

1967

SovietRxiv

Abstract

Full Text

UDC 539.30

THEORY OF ELASTICITY

A. M. VAISMAN, I. A. KUNIN

ON BOUNDARY-VALUE PROBLEMS OF THE CONTINUUM THEORY OF DISLOCATIONS

(Presented by Academician Yu. N. Rabotnov on 17 V 1966)

In this paper Green’ s formula is derived for internal stresses in an elastic anisotropic medium, and the principal boundary-value problems of the continuum theory of dislocations are formulated. An algorithm is indicated for reducing these problems to equivalent basic problems of the ordinary theory of elasticity.

Kröner’ s equations, relating the internal stresses σ and the strain e to the incompatibility η, in direct notation have the form (^{1})

$$\operatorname{div}\sigma = 0, \qquad \sigma = Ce, \qquad \operatorname{Rot} e = \eta, \qquad (\operatorname{div}\eta = 0). \tag{1}$$

Here C is the tensor of elastic constants; the operator Rot in coordinate representation has the form $\eta^{\lambda\nu} = \varepsilon^{\lambda\mu\xi\nu\gamma}\partial_\lambda\partial_\gamma e_{\mu\rho}$, where $\varepsilon^{\lambda\mu}$ is the antisymmetric unit pseudotensor. It is convenient to introduce the stress-function tensor φ, related to σ by the relation $\sigma = \operatorname{Rot}\varphi$. Substituting in (1), we have

$$\operatorname{Rot} B \operatorname{Rot}\varphi = \eta \qquad (B = C^{-1}). \tag{2}$$

In what follows, certain properties of generalized functions concentrated on a surface are used essentially (^{2,3}). The generalized functions $\delta(V)$, $\delta(S)$, and $\delta(L)$, concentrated in the region V, on the surface S, or on the contour L, are defined through the Dirac δ-function as follows:

$$\delta(V) = \int_V \delta(x - x_V)\, dV, \qquad \delta(S) = \int_S \delta(x - x_S)\, dS, \qquad \delta(L) = \int_L \delta(x - x_L)\, dL. \tag{3}$$

If S is the boundary of V, then $\delta(V)$ and $\delta(S)$ are related by

$$\operatorname{grad}\delta(V) = -\delta(S_n), \qquad \operatorname{rot}\delta(V) = -\delta(S_t), \tag{4}$$

where $\delta(S_n) = n\delta(S)$, $\delta(S_t) = t\delta(S)$, n_x is the normal to S, $t^{\lambda\nu} = \varepsilon^{\lambda\varkappa\nu} n_\varkappa$.

We shall assume the solution of equations (2) to be concentrated in the region V. To derive Green' s formula, without loss of generality one may put $\varphi = \varphi_V \delta(V)$, where φ_V is a function sufficiently smooth in a neighborhood of S. The corresponding generalized functions σ and η, with the aid of (4), can be represented in the form

$$\sigma^{\lambda\nu} = \sigma_V^{\lambda\nu}\delta(V) - \delta(S) t^{(\lambda}_{\cdot\varkappa} \operatorname{rot}^{\nu)}_{\cdot\rho} \varphi_V^{\varkappa\rho} - \operatorname{rot}^{(\lambda}_{\cdot\varkappa}[\delta(S) t^{\nu)}_{\cdot\rho} \varphi_V^{\varkappa\rho}], \tag{5}$$

$$\eta = \eta_V \delta(V) - \delta(S_t) \operatorname{rot} e_V - \operatorname{rot}[\delta(S_t) e_V] - \operatorname{Rot} B[\delta(S_t) \operatorname{rot}\varphi_V] - \operatorname{Rot} B \operatorname{rot}[\delta(S_t)\varphi_V]. \tag{6}$$

Here $\sigma_V = Ce_V = \operatorname{Rot}\varphi_V$, $\eta_V = \operatorname{Rot} e_V$. The tensor structure of the terms in the last formula is analogous to the corresponding expressions in (5).

Using the fundamental solution $H^{\alpha\beta}_{\cdot\lambda\nu}(x, x')$ of equations (1) introduced in (^{4}), and taking (6) into account, we obtain Green' s formula for the internal stresses –

stresses

$$\begin{aligned} \sigma(x) = \int_V H(x, x')\,\eta_V(x')\,dV - \int_S H(x, x'_S)\,t(x'_S) \operatorname{rot} e_V(x'_S)\,dS + \\ + \int_S [\operatorname{rot}' H(x, x'_S)]\,t(x'_S) e_V(x'_S)\,dS - \\ - \int_S [B \operatorname{Rot}' H(x, x'_S)]\,t(x'_S) \operatorname{rot}\varphi_V(x'_S)\,dS + \\ + \int_S [\operatorname{rot}' B \operatorname{Rot}' H(x, x'_S)]\,t(x'_S)\varphi_V(x'_S)\,dS. \end{aligned} \tag{7}$$

Here the prime on differential operators means that they act on the second argument. Passing in (7) from the fundamental solution to Green' s function satisfying the corresponding homogeneous conditions on S, one can obtain an integral representation of the solution of the boundary-value problem.

In what follows we consider the case when the sources of internal stresses are dislocations. The density of dislocation moments μ, the dislocation density α, and the incompatibility η are related by the relations (3)

$$\eta^{\lambda\nu} = \operatorname{rot}^{\lambda}_{\cdot\rho}\,\alpha^{\nu\rho} = \operatorname{Rot}^{\lambda\nu}_{\cdot\cdot\kappa\rho}\,\mu^{\kappa\rho}, \qquad \alpha^{\nu\rho} = \operatorname{rot}^{\nu}_{\cdot\kappa}\,\mu^{\kappa\rho}. \tag{8}$$

In particular, to the distribution $\tilde{\mu} = \mu_V \delta(V)$ there correspond the densities

$$\tilde{\alpha}^{\nu\rho} = \alpha_V^{\nu\rho}\delta(V) + \alpha_S^{\nu\rho}\delta(S), \qquad \tilde{\eta}^{\lambda\nu} = \eta_V^{\lambda\nu}\delta(V) - t^{\lambda}_{\cdot\rho}\alpha_V^{\nu\rho}\delta(S) + \operatorname{rot}^{\lambda}_{\cdot\rho}[\alpha_S^{\nu\rho}\delta(S)], \tag{9}$$

where $\eta_V^{\lambda\nu}$ is the volume density of incompatibility, and $\alpha_V^{\nu\rho}$ and $\alpha_S^{\nu\rho}$ are the volume and surface (simple layer) densities of dislocations, defined by the expressions

$$\eta_V^{\lambda\nu} = \operatorname{rot}^{\lambda}_{\cdot\rho}\,\alpha_V^{\nu\rho}, \qquad \alpha_V^{\nu\rho} = \operatorname{rot}^{\nu}_{\cdot\kappa}\,\mu_V^{\kappa\rho}, \qquad \alpha_S^{\nu\rho} = -t^{\nu}_{\cdot\kappa}\mu_V^{\kappa\rho}. \tag{10}$$

Comparing (6) with (9), we find that for η the representation

$$\eta = \tilde{\eta} - \operatorname{Rot} B[\delta(S_t)\operatorname{rot}\varphi_V] - \operatorname{Rot} B\operatorname{rot}[\delta(S_t)\varphi_V] \tag{11}$$

is valid. Obviously, the last two terms in this expression may be interpreted as dislocation layers of higher order.

For the formulation of boundary-value problems let us represent space as filled with two different homogeneous elastic media with interface S. The sources of internal stresses are dislocations distributed inside the first medium, some of the dislocation lines being able to form layers near the interface. It can be shown that, as the elastic constants of the second medium tend to infinity (rigid fixing), the boundary conditions for the first medium follow from the bonding conditions. Let us substitute in (7) the Green tensor $H(x, x')$ of the given boundary-value problem. Then, by virtue of the boundary conditions on H, the last two terms in (7) vanish. Taking (11) into account, we obtain the solution in the form

$$\begin{aligned}\sigma^{\alpha\beta}(x) = \int_V H^{\alpha\beta}_{\cdot\cdot\lambda\nu}(x, x')\,\eta_V^{\lambda\nu}(x')\,dV - \int_S H^{\alpha\beta}_{\cdot\cdot\lambda\mu}(x, x'_S)\,t^{\lambda}_{\cdot\nu}(x'_S)\alpha_V^{\mu\nu}(x'_S)\,dS \\ - \int_S \alpha_S^{\mu\nu}(x'_S)\operatorname{rot}^{\lambda}_{\cdot\nu} H^{\alpha\beta}_{\cdot\cdot\lambda\mu}(x, x'_S)\,dS.\end{aligned} \tag{12}$$

Dislocations concentrated inside V contribute to the volume integral. The first surface integral receives the contribution of those among them whose ends emerge on S. Finally, the last integral receives the contribution of dislocations forming a simple surface layer.

For the continuum theory of dislocations, greater importance attaches to the basic boundary-value problem with a free surface, which is obtained

as the constants of the second medium tend to zero. Assuming, for simplicity, that the dislocation layer of the third order is absent, from the stitching conditions we have

$$t(x_S)\,\mathrm{rot}\,\varphi_V(x_S) = C\mu_S(x_S), \qquad t(x_S)\varphi_V(x_S) = 0, \tag{13}$$

where μ_S is the surface density of dislocation moments. Introducing the corresponding Green tensor and taking into account (7) and (13), we find

$$\sigma^{\alpha\beta}(x) = \int_V H^{\alpha\beta}{}_{\cdot\lambda\nu}(x, x')\eta_V^{\lambda\nu}(x')\,dV - \int_S \mu_S^{\chi\rho}(x'_S)\,\mathrm{Rot}^{\lambda\nu}_{\cdot\chi\rho}\,H^{\alpha\beta}{}_{\cdot\lambda\nu}(x, x'_S)\,dS. \tag{14}$$

In particular, for a single dislocation with contour L and Burgers vector b^λ, to which there corresponds $\alpha^{\nu\rho} = l^\nu b^\rho \delta(L)$ (l^ν is the unit tangent vector to L), the stresses have the form

$$\sigma^{\alpha\beta}(x) = -b^\rho \int_L l^\nu(x'_L)\,\mathrm{rot}^{\lambda}_{\cdot\rho}\,H^{\alpha\beta}{}_{\cdot\lambda\nu}(x, x'_L)\,dL. \tag{15}$$

We note that, when dislocation layers of higher orders are present on S, σ_n undergoes a jump at the boundary, and, consequently, the limit of σ_n from inside for a free boundary is, generally speaking, not equal to zero.

As one of the ways of actually solving the boundary-value problems posed, we indicate the possibility of reducing them to boundary-value problems of the ordinary theory of elasticity. In an unbounded space, a solution of equations (1) can be matched with an equivalent solution of the ordinary theory of elasticity, differing from the first only in the region occupied by the sources (3). Similar equivalent solutions can also be constructed for boundary-value problems. For this purpose let us decompose σ into two parts

$$\sigma = \sigma^0 + \sigma^1 = \mathrm{Rot}[\varphi_V^0\delta(v)] + \mathrm{Rot}[\varphi_V^1\delta(v)], \tag{16}$$

where φ_V^0 is a particular solution of the equation $\mathrm{Rot}\,B\,\mathrm{Rot}\,\varphi_V^0 = \eta_V$. Taking into account (11) and (13) and using the identity $\mathrm{Rot}\,\mathrm{def}[u_V\delta(V)] = 0$, where u_V is a certain vector $^{(1)}$, one can show the validity of the relation

$$\mathrm{Rot}\,B\sigma^1 = \mathrm{Rot}\,\mu_S^*, \qquad C\mu_S^* = C\delta(S_n)u_V - C\delta(S)\mu_S + \delta(S_t)\,\mathrm{rot}\,\varphi_V^0 + {} \\ + \mathrm{rot}[\delta(S_t)\varphi_V^0]. \tag{17}$$

The boundary value u_V entering here, for given α_V, α_S, and φ_V^0, is found by integration over S from the conditions

$$t^\nu{}_\chi \partial^\chi u_V^\rho = \alpha_S^{\nu\rho} - t^\nu{}_\chi\left(e_V^{0\chi\rho} + \Omega^{\chi\rho}\right), \qquad e_V^0 = B\sigma_V^0 = B\,\mathrm{Rot}\,\varphi_V^0, \tag{18}$$

$$2\varepsilon^{\lambda\mu\chi}\varepsilon^{\nu\rho\gamma}n_{[\mu}\partial_{\gamma]}\Omega_{\chi\rho} = t^{\nu}{}_{\chi}\alpha_V^{(\lambda\rho)\chi} - t^{\nu}{}_{\chi}\,\mathrm{rot}^{\lambda}{}_{\cdot\rho}e_V^{0\chi\rho}.$$

The difference $\sigma^* = \sigma^1 - C\mu_S^*$, on the basis of (17), is representable in the form $\sigma^* = C\,\mathrm{def}[u_V\delta(V)]$, and, consequently, σ^* and σ^1 are equivalent in the sense indicated above. With the aid of the identity $\mathrm{div}\,\mathrm{Rot}[\varphi_V^1\delta(V)] = 0$, it can be shown that the generalized force corresponding to σ^* has the form

$$f = -\,\mathrm{div}\,\sigma^0 = \mathrm{div}\,C\mu_S^* = \delta(S_n)[\mathrm{rot}\,C\mu_S - \sigma_V^*] + \mathrm{div}[C\delta(S_n)u_V]. \qquad (19)$$

If α_V and α_S are given, then by formulas (18) one can find the equivalent boundary displacements $u_V(x_S)$ up to a displacement of the region V as a whole. Then, by virtue of (19),

$$\sigma^{*\alpha\beta}(x) = \int_S C^{\lambda\nu}{}_{\cdot\chi\rho}u_V^{\chi}(x'_S)n_{\rho}(x'_S)\partial_{\nu}G^{\alpha\beta}{}_{\cdot\lambda}(x, x'_S)\,dS. \qquad (20)$$

Here

$$G^{\alpha\beta}{}_{\cdot\lambda}(x, x') = C^{\alpha\beta\chi\rho}\partial_{\chi}U_{\rho\lambda}(x, x')$$

is the Green tensor for the stresses of the ordinary theory of elasticity, and $U_{\rho\lambda}$ is the Green tensor for the displacements. For the given problem $U_{\rho\lambda}(x, x'_S) = 0$.

In the case of the second boundary-value problem (13) we have

$$\sigma^{*\alpha\beta}(x) = \int_S G^{\alpha\beta}_{\cdot\lambda}(x, x'_S)\left[\sigma_V^{0\lambda\nu}(x'_S) - \mathrm{rot}'_{\cdot\chi}\,C^{\chi\lambda}_{\cdot\cdot\gamma\varphi}S^{\gamma\varphi}(x'_S)\right]n_{\nu}(x'_S)\,dS, \qquad (21)$$

where G satisfies on S the condition $n_{\beta}(x'_S)G^{\alpha\beta}_{\cdot\lambda}(x'_S, x) = 0$. With the aid of this tensor and the Green tensor H_{∞} for internal stresses in an unbounded medium (^{4}), one can find the Green tensor H for the boundary-value problem with a free surface:

$$H^{\alpha\beta}_{\cdot\lambda\nu}(x, x') = H^{\alpha\beta}_{\infty\cdot\lambda\nu}(x - x') + \int_S G^{\alpha\beta}_{\cdot\chi}(x, x''_S)H^{\chi\rho}_{\infty\cdot\lambda\nu}(x''_S - x')n_{\rho}(x''_S)\,dS. \qquad (22)$$

Here $x, x' \in V$. The last formula is obtained analogously to (21), with the boundary conditions for H being taken into account.

We note that boundary conditions for a medium with dislocations were also considered in (^{1,5}).

Institute of Thermophysics
Siberian Branch of the Academy of Sciences of the USSR

Received
16 V 1966

References

1. E. Kröner, *Kontinuumstheorie der Versetzungen und Eigenspannungen*, 1958.
2. I. M. Gel' fand, G. E. Shilov, *Generalized Functions*, Vol. 1, 1959.
3. I. A. Kunin, *Prikl. Mekh. i Tekhn. Fiz.*, **5**, 76 (1965).
4. I. A. Kunin, *DAN*, **157**, 1319 (1964).
5. G. Rieder, *Österreich. Ing. Arch.*, **18**, H. 3–4 (1964).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.