ON AN EXTREMAL PROBLEM OF THERMOELASTICITY FOR AN INFINITE CYLINDRICAL SHELL

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Abstract

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ON AN EXTREMAL PROBLEM OF THERMOELASTICITY FOR AN INFINITE CYLINDRICAL SHELL

A variational problem is considered for determining axisymmetric temperature fields in an infinite cylindrical shell that ensure a minimum of the functional of the elastic strain energy under prescribed conditions in fixed cross sections. A special case of the obtained solution is investigated as applied to conditions of local heating.

1. Let a free infinite isotropic cylindrical shell of radius R be under the action of an axisymmetric temperature field that is constant over the thickness and vanishes at infinity. The thermoelastic state of the shell is characterized by the circumferential force N and the axial moment M, which are determined through the temperature T and the dimensionless deflection $w_0 = w/R$ by the formulas ([1,2])

$$N = 2Eh(w_0 - \alpha T), \qquad M = -\frac{EhR}{2a^2}\frac{d^2 w_0}{d\xi^2}, \tag{1,1}$$

where $\xi = az/R$; $a^4 = 3(1-\nu^2)R^2/4h^2$; $2h$ is the shell thickness; E is the modulus of elasticity; ν is Poisson' s ratio; α is the linear coefficient of thermal expansion; z is the axial coordinate. The deflection function w_0 must satisfy the resolving equation

$$d^4 w_0/d\xi^4 + 4(w_0 - \alpha T) = 0 \tag{1,2}$$

and tend to zero as $\xi \to \infty$.

The elastic energy of the shell, taking (1,1), (1,2) into account, is written in the form

$$K[w_0] = \frac{\pi EhR^2}{8a}\int_{-\infty}^{\infty}\left[\left(\frac{d^4 w_0}{d\xi^4}\right)^2 + 4\left(\frac{d^2 w_0}{d\xi^2}\right)^2\right] d\xi, \tag{1,3}$$

i.e., it is a functional defined on the set of functions $w_0 = w_0(\xi)$.

The following variational problem is formulated. Find the extremum of the functional $K[w_0]$ on the set of functions $w_0 = w_0(\xi)$ continuous together with their derivatives up to the third order inclusive and vanishing at infinity, which satisfy, in fixed sections $\xi = \xi_j$ $(j = 1, 2, \dots, n)$, the conditions

$$d^{(i)} w(\xi_j)/d\xi^i = w_{ij} \qquad (i = 0, 1, 2, 3), \tag{1,4}$$

where w_{ij} are arbitrary numbers that can be determined by prescribing, in the sections $\xi = \xi_j$, numerical values of the problem parameters (deflection, temperature, circumferential force, moment).

The formulated problem is equivalent to the following isoperimetric problem. Find the extremum of the functional $K[w_0]$ on the set of continuous and four-times continuously differentiable functions $w_0 = w_0(\xi)$, on which the singular functionals

$$K_{ij}[w_0] = (-1)^i \int_{-\infty}^{\infty} \delta^{(i)}(\xi - \xi_j) w_0(\xi)\, d\xi, \tag{1,5}$$

where $\delta^{(i)}(\xi)$ is the i-th derivative of the delta function, take the prescribed values

$$K_{ij}(w_0) = w_{ij}. \tag{1,6}$$

Such a problem reduces to finding the absolute extremum of the functional ([3])

$$K^*[w_0] = \frac{\pi E h R^2}{8a} \int_{-\infty}^{\infty} \left[\left(\frac{d^4 w_0}{d\xi^4} \right)^2 + 4 \left(\frac{d^2 w_0}{d\xi^2} \right)^2 - 2 w_0(\xi) \sum_{i=0}^{3} \sum_{j=0}^{n} \gamma_{ij} \delta^{(i)}(\xi - \xi_j) \right] d\xi. \tag{1,7}$$

Here γ_{ij} are arbitrary constants ensuring satisfaction of conditions (1,4).

The Euler equations for the functional $K^*[w_0]$ give

$$\frac{d^8 w_0}{d\xi^8} + 4 \frac{d^4 w_0}{d\xi^4} = \sum_{i=0}^{3} \sum_{j=1}^{n} \gamma_{ij} \delta^{(i)}(\xi - \xi_j). \tag{1,8}$$

In view of (1,2), from (1,8) we find

$$\frac{d^4 T}{d\xi^4} = \frac{1}{4a} \sum_{i=0}^{3} \sum_{j=0}^{n} \gamma_{ij} \delta^{(i)}(\xi - \xi_j). \tag{1,9}$$

Equations (1,2) and (1,8), or (1,9), together with the conditions at infinity constitute the complete system of relations for determining the extremal temperature distribution and the corresponding displacement $w_0(\xi)$.

2. The solution of equations (1,2) and (1,8), satisfying the conditions at infinity, has the form

$$T(\xi) = \frac{1}{8a}\sum_{j=1}^{n}\left[\frac{\gamma_{0j}}{6}(\xi-\xi_j)^3 + \frac{\gamma_{1j}}{2}(\xi-\xi_j)^2 + \gamma_{2j}(\xi-\xi_j) + \gamma_{3j}\right]\operatorname{sgn}(\xi-\xi_j); \tag{2,1}$$

$$\begin{aligned} w_0 = \frac{1}{8}\sum_{j=1}^{n}\Bigg\{ &\left[\frac{\gamma_{0i}}{6}(\xi-\xi_j)^3 + \frac{\gamma_{1j}}{2}(\xi-\xi_j)^2 + \gamma_{2j}(\xi-\xi_j) + \gamma_{3j}\right]\operatorname{sgn}(\xi-\xi_j) \\ &+ e^{-|\xi-\xi_j|}\left[-\frac{\gamma_{0j}}{4}(\cos(\xi-\xi_j) + \sin|\xi-\xi_j|) + \frac{\gamma_{1j}}{2}\sin(\xi-\xi_j)\right. \\ &\left.+\frac{\gamma_{2j}}{2}(\cos(\xi-\xi_j) - \sin|\xi-\xi_j|) - \gamma_{3j}\operatorname{sgn}(\xi-\xi_j)\cos(\xi-\xi_j)\right]\Bigg\}. \end{aligned} \tag{2,2}$$

In this case the coefficients γ_{ij} must satisfy the relations

$$\sum_{j=1}^{n}\gamma_{0j} = 0, \qquad \sum_{j=1}^{n}(\gamma_{0j}\xi_j - \gamma_{1j}) = 0, \qquad \sum_{j=1}^{n}(\gamma_{0j}\xi_j^2 - 2\gamma_{1j}\xi_j + \gamma_{2j}) = 0,$$

$$\sum_{j=1}^{n}(\gamma_{0j}\xi_j^3 - 3\gamma_{1j}\xi_j^2 + 6\gamma_{2j}\xi_j - 6\gamma_{3j}) = 0. \tag{2,3}$$

The circumferential force N and the axial moment M, calculated by formulas (1.1) with allowance for (2.1), (2.2), will be

$$\begin{aligned} N = \frac{Eh}{16}\sum_{j=1}^{n}\big[&-\gamma_{0j}\left(\cos(\xi-\xi_j) + \sin|\xi-\xi_j|\right) + 2\gamma_{1j}\sin(\xi-\xi_j) + \\ &+2\gamma_{2j}\left(\cos(\xi-\xi_j) - \sin|\xi-\xi_j|\right) - 4\gamma_{3j}\operatorname{sgn}(\xi-\xi_j)\cos(\xi-\xi_j)\big]\, e^{-|\xi-\xi_j|}; \end{aligned} \tag{2.4}$$

$$\begin{aligned} M = -\frac{EhR}{32a^2}\sum_{j=1}^{n}\big\{&2\gamma_{0j}|\xi-\xi_j| + \gamma_{0j}\left(\cos(\xi-\xi_j) - \sin|\xi-\xi_j|\right)e^{-|\xi-\xi_j|} + \\ &+2\gamma_{1j}\left(1 - e^{-|\xi-\xi_j|}\cos(\xi-\xi_j)\right)\operatorname{sgn}(\xi-\xi_j) + \big[2\gamma_{2j}\left(\cos(\xi-\xi_j) + \right. \end{aligned}$$

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

$$+\sin|\xi-\xi_j|\big) - 4\gamma_{3j}\sin(\xi-\xi_j)\big]\, e^{-|\xi-\xi_j|}\Big\}\,. \tag{2.5}$$

From formulas (2.1)–(2.5) it is evident that the experimental temperature distribution T found and the force N are described by piecewise-continuous functions. A distribution of T and N continuous in ξ is obtained by setting $\gamma_{3j}=0$. If one also requires continuity of the first derivative, it is necessary in addition to set $\gamma_{2j}=0$.

Let us consider the particular case of a solution as applied to boundary conditions of local heating of the simplest form. Let the zone of local heating of the cylindrical shell be bounded by the sections $\xi=\pm\eta$. The temperature in the end sections ($\xi=\pm\eta$) is equal to zero. In the section $\xi=0$ the temperature T reaches its maximum value, equal to T_0.

Fig. 1

Fig. 2

The temperature field (2.1) that is extremal for such a problem, satisfies the condition of symmetry with respect to the section $\xi=0$ and is continuous together with its first derivative, is the field

$$T = T_0\left[2|\xi/\eta|^3 - 3(\xi/\eta)^2 + 1\right] \quad \text{for } |\xi|\le\eta, \qquad T=0 \quad \text{for } |\xi|\ge\eta. \tag{2.6}$$

The circumferential force and axial moment corresponding to (2.6) are determined by the formulas

$$\begin{aligned} N = \frac{3Eh\alpha T_0}{\eta^3}\,\big[&(\cos(\xi+\eta)+\sin|\xi+\eta|)e^{-|\xi+\eta|} + (\cos(\xi-\eta)+ \\ &+\sin|\xi-\eta|)e^{-|\xi-\eta|} - 2(\cos\xi+\sin|\xi|)e^{-|\xi|}+ \\ &+\eta\left(e^{-|\xi+\eta|}\sin(\xi+\eta) - e^{-|\xi-\eta|}\sin(\xi-\eta)\right)\big]\,, \end{aligned} \tag{2.7}$$

$$M = \frac{3EhR\alpha T_0}{2a^2\eta^3}\Big[2|\xi+\eta| + 2|\xi-\eta| - 4|\xi|$$
$$+ (\cos(\xi+\eta) - \sin|\xi+\eta|)e^{-|\xi+\eta|} + (\cos(\xi-\eta) - \sin|\xi-\eta|)e^{-|\xi-\eta|}$$
$$- 2(\cos\xi - \sin|\xi|)e^{-|\xi|} \qquad (2,8)$$
$$-\eta(1-\cos(\xi+\eta)e^{-|\xi+\eta|})\,\mathrm{sgn}(\xi+\eta) + \eta(1-\cos(\xi-\eta)e^{-|\xi-\eta|})\,\mathrm{sgn}(\xi-\eta)\Big].$$

In Fig. 1 the graphs of $N^* = N/Eh\alpha T_0$ and $M^* = a^2M/EhR\alpha T_0$ are presented for $\nu = 0.3$; $R/h = 20$; 40; $\eta = a$. In Fig. 2 the graphs are given of the quantity N^* in the section $\xi = 0$ and at the two nearest points $\xi = \xi^{(1)}$ and $\xi = \xi^{(2)}$, where N^* attains an extremal value, for values of η in the range $1 \leqslant \eta \leqslant 6$.

The found quantities $\xi^{(1)}$ and $\xi^{(2)}$ for several values of η have the following values.

η	1	2	3	4	5	6
$\xi^{(1)}$	0,951	1,74	2,52	3,40	4,37	5,35
$\xi^{(2)}$	3,27	3,67	4,31	5,12	6,03	6,98

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Note: Figure translations are in progress. See original paper for figures.