A NEIGHBORHOOD OF A SINGULAR POINT OF A SYSTEM OF DIFFERENTIAL EQUATIONS WITH SMALL NONLINEARITIES

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Abstract

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D. M. GROBMAN

A NEIGHBORHOOD OF A SINGULAR POINT OF A SYSTEM OF DIFFERENTIAL EQUATIONS WITH SMALL NONLINEARITIES

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1°. Consider the systems

$$x' = Ax + F(t, x), \tag{1}$$

$$y' = Ay, \tag{2}$$

where A is a constant $n \times n$-matrix; x, y, F are n-dimensional vectors. Suppose that

$$F(t, 0) = 0 \tag{3}$$

and in the domain $|x| \le r_0 < 1$ the condition

$$|F(t, x_1) - F(t, x_2)| \le L(r)|x_1 - x_2|, \tag{4}$$

is satisfied, where $r = \max(|x_1|, |x_2|)$; $L(r)$ decreases monotonically as $r \to 0$.

In note ([1]), under the assumption that the vector $F(t, x)$ is given only for $t \ge 0$, several propositions were formulated concerning the asymptotic behavior of solutions of system (1) as $t \to +\infty$.

If one assumes that the vector $F(t, x)$ is defined for all t and has properties (3) and (4), then the results of ([1]) can be strengthened by considering the behavior of solutions of system (1) both as $t \to +\infty$ and as $t \to -\infty$.

There are propositions analogous to Theorems 1-3 of note ([1]). In order not to repeat ourselves, we shall give the formulation of only one of them, for example the analogue of Theorem 3 from ([1]).

2°. Let us recall some definitions needed for the formulation of the theorem.

The **exponent of the vector** $x(t)$ is the number

$$\chi(x) = \overline{\lim_{t\to+\infty}} \frac{1}{t} \ln|x(t)|.$$

The **minus-exponent** $\bar{\chi}(x)$ of the vector $x(t)$ is defined by the equality

$$\bar{\chi}(x) = \overline{\lim_{t\to-\infty}} \frac{1}{-t} \ln|x(t)|.$$

The exponent and minus-exponent characterize the exponential growth of $x(t)$ as $t \to +\infty$ and $t \to -\infty$.

For a more precise estimate of the growth of $x(t)$, introduce other characteristics. The **second exponent** of the vector $x(t)$ with exponent ω is the number

$$\chi_2(x) = \overline{\lim_{t\to+\infty}} \frac{\ln\left(e^{-\omega t}|x(t)|\right)}{\ln t}.$$

The **second minus-exponent** $\bar{\chi}_2(x)$ of the vector $x(t)$ with minus-exponent ω will be defined by the formula

$$\bar{\chi}_2(x) = \overline{\lim_{t\to-\infty}} \frac{\ln\left(e^{-\omega|t|}|x(t)|\right)}{\ln|t|}.$$

The ratio $|x(t)-y(t)|/|y(t)|$ will be called the **deviation of the vector** x **from the vector** y. If the deviation of x from y tends to zero as $t \to +\infty$ ($t \to -\infty$), then we shall say that x and y are **analogous as** $t \to +\infty$ ($t \to -\infty$).

3°. Introduce the following notation: $\omega_1 < \omega_2 < \dots < \omega_s$ are the distinct real parts of the eigenvalues of the matrix A; $m_k + 1$ is the maximum of the orders of those boxes in the Jordan form of the matrix A which correspond to the number ω_k. We shall assume that $\omega_1 < 0,\ \omega_s > 0$, since the case $\omega_1\omega_s > 0$ was essentially considered in note (1).

Let the function $L(r)$ be given by the equalities

$$L(0) = 0, \qquad L(r) = \varepsilon(r) r^{\lambda} |\ln r|^{\mu}, \qquad r \in (0, r_0], \tag{5}$$

where $\lambda > 0$, μ is an arbitrary number, $\varepsilon(r) \geq 0$ increases monotonically together with r, and

$$\int_0^{r_0} \frac{\varepsilon(r)}{r|\ln r|} < +\infty. \tag{6}$$

Consider the numbers $\omega_0 < \omega_1(1+\lambda)$, $\omega_{s+1} > \omega_s(1+\lambda)$, and for each $k = 1, 2, \dots, s$ find an index $\tilde{k}$ from the inequalities

$$\omega_{\tilde{k}-1} < (1+\lambda)\omega_k \leq \omega_{\tilde{k}}, \quad \text{if } \omega_k \leq 0, \tag{7}$$

$$\omega_{\tilde{k}-1} \leq (1+\lambda)\omega_k < \omega_{\tilde{k}}, \quad \text{if } \omega_k > 0. \tag{8}$$

By $m_{\tilde{k}}^0$ denote the number which is equal to 0 if the inequality (7) or (8) corresponding to the index k is strict, and equal to m in the contrary case.

Define the function $\rho(t)$ by the equalities

$$\rho(t) = \begin{cases} 1, & \text{for } |t| \leq 1, \\ |t|, & \text{for } |t| > 1. \end{cases}$$

4°. Theorem. *Suppose that conditions (3)–(6) are satisfied. Then:*

1) Every solution $x(t)$ of system (1) with exponent (minus-exponent) $\omega < 0$ is analogous to some solution $y(t)$ of system (2), and the deviation of x from y is

$$o\left(e^{\lambda\omega|t|}|t|^{m_{\tilde{k}}{}^0+1+\lambda l+\mu}\right) \quad \text{as } t \to +\infty \quad (t \to -\infty), \tag{9}$$

where l is the second exponent (second minus-exponent) of x and y.

2) For every solution $x(t)$ of system (1) which at $t = 0$ passes through a sufficiently small neighborhood S^* of the origin and has a negative exponent (minus-exponent) ω, the following inequality holds for $t \geq 0$ $(t \leq 0)$:

$$|x(t)| \leq M|x(0)|e^{\omega|t|}\rho^l(t);$$

$M > 0$ does not depend on x; l is the second exponent (second minus-exponent) of x.

3) There exists a homeomorphism Φ^*, mapping the neighborhood S^* onto some domain G, possessing the following properties:

a) $\Phi(0) = 0$;

b) if $y(0) \in G$ and the solution $y(t)$ of system (2) has exponent (minus-exponent) $\omega < 0$, then through the point $\Phi^{*-1}(y(0))$ at $t = 0$ there passes a solution $x(t)$ of system (1), analogous to $y(t)$ as $t \to +\infty$ $(t \to -\infty)$ and with deviation (9); moreover, for $t \geq 0$ $(t \leq 0)$,

$$|x(t) - y(t)| \leq |x(0)|^{1+\lambda}\psi(t, |x(0)|)e^{(1+\lambda)\omega|t|}\rho(t)^{m_{\tilde{k}}{}^0+(1+\lambda)l+1+\mu}, \tag{10}$$

and for $\mu \le 0$

$$\psi(t,r) \to 0, \quad \text{when } |t| + r^{-1} \to \infty,$$

for $\mu > 0$

$$\psi(t,r) = |\ln r|^{\mu} \varepsilon(t,r),$$

where $\varepsilon(t,r) \to 0$ when $|t| + r^{-1} \to \infty$;

c) the homeomorphisms Φ^* and Φ^{*-1} satisfy the Lipschitz condition and have the form

$$\Phi^*(x) = x + \varphi^*(x), \qquad \Phi^{*-1}(x) = x + \psi^*(x),$$

where

$$|\varphi^*(x)| = o(|x|^{1+\lambda}) \quad \text{as } x \to 0, \quad \text{if } \mu \le 0,$$

$$|\varphi^*(x)| = o(|\ln|x||^{\mu}|x|^{1+\lambda}) \quad \text{as } x \to 0, \quad \text{if } \mu > 0;$$

$$|\psi^*(x)|/|\varphi^*(x)| \to 1, \qquad \text{when } x \to 0.$$

4) If none of the numbers $\omega_i \ne 0$ $(i = 1, 2, \dots, s)$, then through the points corresponding, under the mapping Φ^*, to $t = 0$ there pass either saddle solutions $x(t)$ and $y(t)$ of systems (1) and (2), or 0-curves; in the latter case all assertions of item 3 b) are valid for x and y.

Institute of Electronic
Control Machines

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CITED LITERATURE

1. D. M. Grobman, DAN, **166**, No. 1, 15 (1966).

Note: Figure translations are in progress. See original paper for figures.