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Abstract

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ON THE EXTENSION OF A. N. KOLMOGOROV' S UNIFORM THEOREM TO HOMOGENEOUS MARKOV CHAINS

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In the works ([1],[2]) A. N. Kolmogorov gave an estimate for the approximation of the distributions of sums of independent, identically distributed summands by infinitely divisible distributions. In the present note this result is extended to the case of distributions of sums of summands connected in a homogeneous Markov chain.

Let, on a probability space $\{\Omega, \mathcal{F}, P\}$, a sequence of random variables $\{X_k = X_k(\omega)\}$ be given, connected in a homogeneous Markov chain; Ω is an arbitrary set of states ω; $\mathcal{F}$ is a σ-algebra of its subsets A; P is a probability measure of the elements of the set $\mathcal{F}$. Denote by $P_{ij}(\omega, A)$ the probabilistic transition function in $j-i$ steps;

$P(A)$ is the probability of A; $c, c_1, c_2, \ldots$ are absolute constants; $s_n = \sum_{i=1}^{n} X_i$,

$$F_n = P(s_n < x), \quad F = P(X_i < x).$$

We shall assume that the ergodic property of the chain is characterized by the condition adopted in the works of S. V. Nagaev ([3]):

$$\sup_{\omega, A} \left|P_{ij}(\omega, A) - P(A)\right| \le Cq^{\mu}, \quad 0 < q < 1, \quad \mu = j - i.$$

We note that the ergodicity condition used in the works of R. L. Dobrushin ([4]) would not change the subsequent reasoning.

Theorem. Let G be the set of all infinitely divisible distribution laws $D(x)$. Then there exists an absolute constant C_1 such that

$$\rho(F_n, G) = \sup_F \inf_{D_n \in G} \sup_x |F_n - D_n| \leq c_1 n^{-1/3} \ln^{\lambda/3} n, \tag{1}$$

where $\lambda > 1$.

We outline the proof. Using the idea of S. N. Bernstein ([5]), we construct sums of independent random variables by whose distributions we shall approximate F_n. Write s_n as follows:

$$s_n = y_1 + u_1 + \ldots + y_l + u_l,$$

where

$$y_i = X_{(i-1)(r+k)+1} + \ldots + X_{i(r+k)-k}; \quad u_i = X_{i(r+k)-k+1} + \ldots + X_{i(r+k)};$$

$$i = 1, \ldots, l; \quad (r+k)l \geq n; \quad \Sigma_l = y_1 + \ldots + y_l; \quad \sigma_l = u_1 + \ldots + u_l;$$

$$s_n = \Sigma_l + \sigma_l.$$

Put $k = r = \ln^\lambda n,\ \lambda > 1$. Then $l \geq n/2 \ln^\lambda n$.

Denote

$$F_\Sigma(x) = P(\Sigma_l < x), \quad F_\sigma(x) = P(\sigma_l < x),$$

and $F_\Sigma(x/A)$, $F_\sigma(x/A)$ the corresponding conditional distribution functions (d.f.) under some condition A; Φ_k' is some conditional d.f., and Φ_k the unconditional d.f. of y_k.

The rest of the proof rests on three lemmas.

Lemma 1. Let G_l and f be the characteristic functions (c.f.) of Σ_l and y_k, respectively. Then

$$G_l - f^l = \sum_{r=1}^{l} \delta_r L_r, \quad \text{where} \quad \delta_r = \int e^{itx}\, d[\Phi_r' - \Phi_r], \quad L_r \equiv G_{r-1} f^{l-r}.$$

Proof is carried out in many respects in the same way as that of Lemma 2 in the author' s paper ([6]).

Lemma 2. One can indicate such a continuous d.f. $\Psi_\Sigma(x)$ of the sum of l terms, related to one another in the same way as the y_k in the sum Σ_l, that

$$|F_\Sigma(x) - \Psi_\Sigma(x)| \leq c_2 n^{-1/3}.$$

For the proof we compose $\Phi(x/\varepsilon)$—a certain normal distribution—with $F_\Sigma(x)$. We show that

$$|F_\Sigma(x) - \Psi_\Sigma(x)| \leq |\Psi_\Sigma(x-2h\varepsilon) - \Psi_\Sigma(x+2h\varepsilon)| + \frac{2c_3}{h} e^{-h^2/2}, \qquad \Psi_\Sigma = F_\Sigma * \Phi.$$

The estimate

$$\alpha \equiv |\Psi_\Sigma(x-2h\varepsilon) - \Psi_\Sigma(x+2h\varepsilon)|$$

is carried out by S. N. Bernstein' s lemma $((7), p.409)$:

$$\alpha \leq \frac{2\varepsilon}{\pi}\left[A \cdot 2h\varepsilon + \frac{1}{2\sqrt{Nh\varepsilon}}\right],$$

$$A = \frac{1}{2}\left|\int_{-N}^{N} \varphi_l(t)\, dt\right|;$$

φ_l is the ch.f. for the d.f. $\Psi_\Sigma(x)$. Then, by Lemma 1,

$$A = \frac{1}{2}\left|\int_{-N}^{N} f^l(t)\, dt + \sum_{k=1}^{l} \int_{-N}^{N} \delta_k L_k\, dt\right|.$$

The first integral is estimated by known methods, while the second is reduced to the Dirichlet integral. Setting $h = \ln^2 n$, $\varepsilon = \ln^{-2} n \cdot n^{-1/3}$, $\sqrt[3]{n} = n^{2/3}$, and using the ergodicity condition, we verify the validity of the lemma.

Lemma 3. Let $\bar{F}_\Sigma(x)$ denote the d.f. of the sum $y_1 + \cdots + y_l$ under the assumption of continuity and independence of the terms. Then

$$|\bar{F}_\Sigma(x) - \Psi_\Sigma(x)| \leq c_4 n^{-1/3}.$$

Proof. For the d.f.' s $\bar{F}_\Sigma(x)$ and $\Psi_\Sigma(x)$, the corresponding ch.f.' s can be written as follows: $f^l\varphi; G_l\varphi$, where φ is the ch.f. for $\Phi(x/\varepsilon)$. Next we use Esseen' s theorem ([8]), for the application of which we consider

$$\Delta = \left|\int_{-T}^{T} e^{-itx} \frac{(f^l - G_l)\varphi}{-it}\, h(t)\, dt\right|$$

instead of

$$\int_{-T}^{T} \frac{|f^l - G_l|\,|\varphi|}{|t|}\,dt,$$

and verify the validity of the lemma.

Further,

$$F_n(x) = M\left\{\int F_\Sigma\left(\frac{x-t}{A}\right)\,dF_\sigma\left(\frac{t}{A}\right)\right\}.$$

Since Lemma 2 remains valid also for conditional distributions, i.e.

$$\left|F_\Sigma\left(\frac{x-t}{A}\right) - \Psi_\Sigma\left(\frac{x}{A}\right)\right| \le c_2 n^{-1/3},$$

the error in replacing $F_\Sigma((x-t)/A)$ by a continuous function will not exceed $c_2 n^{-1/3}$. We shall assume, without making elementary explanations for simplicity, that $F_\Sigma((x-t)/A)$ is continuous. From

$$F_n(x) = M\left[\int_{-\infty}^{\xi} + \int_{\xi}^{\infty}\right]$$

we obtain

$$F_\sigma(\xi) - 1 \le F_n(x) - F_\Sigma(x-\theta) \le 0,$$

$-\infty < \theta \le \xi$. We choose ξ so that $1 - F_\sigma(\xi) \le n^{-1/3}$. Next we replace $F_\Sigma(x-\theta)$ by the distribution $\bar{F}_\Sigma(x-\theta)$. Then, using Lemma 3, we obtain

$$-c_5 n^{-1/3} \le F_n(x) - \bar{F}_\Sigma(x-\theta) \le c_5 n^{-1/3}. \qquad (2)$$

According to A. N. Kolmogorov' s theorem ([2]), there exists an infinitely divisible distribution $D_n(x)$ such that

$$|\bar{F}_\Sigma(x-\theta) - D_n(x)| \le c_6 n^{-1/3} \ln^\lambda n,$$

since the number of terms in Σ_l is equal to $l \ge n/2\ln^\lambda n$. Combining this with (2), we arrive at the conclusion that (1) holds. The theorem is proved.

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References

1. A. N. Kolmogorov, *Theory of Probability and Its Applications*, 1, § 4, 426 (1956).

2. A. N. Kolmogorov, *Trudy Moskov. matem. obshch.*, 12, 437 (1963).

3. S. V. Nagaev, *Theory of Probability and Its Applications*, 2, No. 4, 389 (1957).

4. R. L. Dobrushin, *Theory of Probability and Its Applications*, 1, No. 1, 72 (1956).

5. S. N. Bernstein, *UMN*, 10 (1944).

6. D. V. Manevich, *Collected Papers: Probability Theory and Mathematical Statistics*, vol. 1, Academy of Sciences of the Uzbek SSR, 1964.

7. S. N. Bernstein, *Probability Theory*, Moscow, 1946.

8. C. G. Esseen, *Acta Math.*, 77, 1 (1945).

Note: Figure translations are in progress. See original paper for figures.