The existence and uniqueness theorem for the Cauchy problem of a hyperbolic equation with self-regulating delay

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Abstract

For a hyperbolic equation of the form

$$U_{tx} = f(t, x, U(t, x), U(t - \tau, x), U_t(t, x), U_t(t - \tau, x), U_x(t, x)U_x(t - \tau, x))$$

with an initial function $\varphi(t, x)$ defined for $(t, x) \in [t_0 - \tau_0, t_0] \times \Omega$ and with a delay $\tau = \tau(t, x, U, U_t, U_x)$ that depends not only on the independent variables x, t but also on the unknown function and its derivatives—referred to as a self-regulating delay—a local existence and uniqueness theorem for the solution to the Cauchy problem is established using the contraction mapping principle. It is assumed that the functions τ, f, φ are continuous in the aggregate of their arguments, and the functions f and τ, moreover, satisfy Lipschitz conditions with respect to their arguments starting from the third, uniformly with respect to t and x. 1 figure. 6 references.

Full Text

Preamble

This section considers the existence and uniqueness of solutions for a class of partial differential equations with functional arguments. Specifically, we investigate the following equation:

$$U_{xt} = f(t, x, U(t, x), U(t - \tau, x), U_t(t, x), U_t(t - \tau, x), U_x(t, x), U_x(t - \tau, x))$$

where the delay argument is defined by $\tau = \tau(t, x, U(t, x), U_x(t, x), U_t(t, x))$. The domain of interest is defined for $t \in [t_0 - \tau_0, t_0 + h]$ and $x \in \Omega$. We assume that the delay satisfies the condition $\tau(t, x, U, U_x, U_t) < t$, ensuring that the functional dependence remains within the past state of the system. This work builds upon the foundational results established in [?, ?] and further developed in [?, ?].

1. Existence and Uniqueness Conditions

To establish the existence of a solution $U(t,x)$ for equation (1), we impose the following conditions:

a) The initial function $\phi(t,x)$ is defined and sufficiently smooth on the initial set $[t_0 - \tau_0, t_0] \times \Omega$. We assume that ϕ and its derivatives $\phi_t, \phi_x, \phi_{tx}$ are continuous and bounded.

b) The function f satisfies a Lipschitz condition with respect to its functional arguments. Specifically, there exists a constant $L_f > 0$ such that for any two sets of arguments, the difference in f is bounded by the sum of the absolute differences of its components.

c) The delay function τ is also Lipschitz continuous with constant $L_\tau > 0$. We assume that for $t \in [t_0, t_0 + h]$, the solution remains within a bounded region where the partial derivatives of f and τ are controlled.

2. Operator Formulation and Convergence

The problem can be reformulated as a fixed-point problem for an operator T acting on a function space V. We define the operator $T(U(t,x))$ based on the integral form of the differential equation:

$$T(U(t,x)) = \phi(t_0, x) + \int_{t_0}^{t} \int_{x_0}^{x} f(\xi, \eta, U, U_\tau, U_t, U_{t\tau}, U_x, U_{x\tau})\, d\eta\, d\xi$$

To prove the existence of a unique solution, we demonstrate that T is a contraction mapping in a suitably chosen Banach space. We define the norm on this space as:

$$\rho(V,W) = \sup |V(t,x) - W(t,x)| + \sup |V_t(t,x) - W_t(t,x)| + \sup |V_x(t,x) - W_x(t,x)|$$

By applying the Lipschitz conditions on f and τ, we derive an estimate for $\rho(T(V), T(W))$. Let B_1 and B_2 be bounds such that $|f| \leq B_1$ and the partial derivatives of f are bounded by B_2. After a series of estimations over the domain ABC (as illustrated in [Figure 1: see original paper]), we obtain:

$$\rho(T(V), T(W)) \leq L_f \cdot h \cdot [2 + (B_1 + B_1 a_1 + B_2 a_1) L_\tau] \cdot \rho(V,W)$$

For the operator to be a contraction, we require the coefficient $\alpha = L_f \cdot h \cdot [2 + (B_1 + B_1 a_1 + B_2 a_1) L_\tau]$ to be less than 1. This condition is satisfied by choosing a sufficiently small time interval h. Under these constraints, the Banach fixed-point theorem guarantees the existence of a unique solution $U(t,x)$ for the given initial conditions.

References

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Note: Figure translations are in progress. See original paper for figures.