GENERATORS FOR PERFECT PARTITIONS

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Abstract

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GENERATORS FOR PERFECT PARTITIONS

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Introduction. V. A. Rokhlin and Ya. G. Sinai ([1]) introduced into ergodic theory the notion of a perfect partition as the most natural analogue of a partition generated by the past of a stationary random process with a finite number of states. Rokhlin ([2]) showed that an ergodic automorphism of a Lebesgue space having finite entropy possesses a generator with finite entropy and, consequently, can be represented as a stationary (in the strict sense) random process whose state space is countable and has finite entropy. The problem of the existence of a countable generator for an arbitrary ergodic automorphism of a Lebesgue space was solved by Rokhlin ([3]). Somewhat later the same result was obtained by another method by the author of the present article (see ([6])). This problem was also mentioned by Sinai ([4]), who in ([5]) touched upon the related problem of the existence of a countable generator for a perfect partition, subsequently also solved by Rokhlin ([3]). Rokhlin showed that a perfect partition can be generated by a partition with finite entropy if the automorphism has finite entropy.

In this article we consider an ergodic automorphism with finite entropy. We show that a perfect partition can be generated by a countable partition which is a strong generator (see ([6])) for the inverse automorphism (the generator has, generally speaking, infinite entropy). This result can also be derived from ([3]). Here we propose a more direct route to the proof.

For K-automorphisms our result means that the automorphism can be represented as a process with a countable number of states, and the process itself is regular, while the process obtained from it by reversing time is singular (sometimes, instead of regular, one says completely nondeterministic, and instead of singular—deterministic). An analogous effect in one special example was communicated to me by D. A. Friedman. This circumstance, as I. N. Dowker suggested, might prevent a strong generator from being a generator for a perfect partition. However, from our main result it follows that this is not so.

The distinction between countable partitions and countable partitions with finite entropy is decisive, and one may suppose that Sinai (see ([4,5,7])), speaking in certain cases about countable partitions, had in mind partitions with finite entropy.

§ 1. **Notation.** The basic properties of a Lebesgue space, measurable partitions, etc., are set out in ([8]). For the definition of the entropy $H(\xi)$ and the mean conditional entropy $H(\xi \mid \eta)$ for arbitrary measurable partitions ξ and η, see ([2,3]). Throughout, T will denote an ergodic automorphism of a Lebesgue space with continuous measure (X, B, m). If ξ is a measurable partition such that $T^{-1}\xi \leq \xi$, then T_ξ will denote the factor-endomorphism of the space (X_ξ, B_ξ, m_ξ), i.e. $(X_\xi, B_\xi, m_\xi, T_\xi)$ is determined by the condition that the mapping

$$\Phi : (X, B, m, T) \to (X_\xi, B_\xi, m_\xi, T_\xi),$$

where $X_\xi = \xi$, $\Phi(x) = c$ for $x \in c \in \xi$, is a homomorphism. Let Z denote the set of partitions ξ with $H(\xi) < \infty$. The elements of Z (Z-partitions) are necessarily countable, but, of course, there exist countable partitions not belonging to Z.

We write

$$\xi_T^n = \xi^n = \bigvee_{k=0}^{n} T^k\xi, \qquad \xi_T^- = \xi^- = \bigvee_{k=0}^{\infty} T^{-k}\xi, \qquad \xi_T^+ = \xi^+ = \bigvee_{k=0}^{\infty} T^k\xi,$$

$$\xi_T = \bigvee_{k=-\infty}^{\infty} T^k\xi = \xi_T^- \vee \xi_T^+,$$

$$h(T, \xi) = H(T\xi \mid \xi_T^-), \qquad h(T) = \sup_{\xi \in Z} h(T, \xi);$$

$h(T)$ is the entropy of T. By ε we shall denote the partition of the space X into individual points, and by ν the partition whose only element is all of X.

A partition ξ is called a **generator** (**strong generator**) if $\xi_T = \varepsilon$ ($\xi^- = \varepsilon$); ζ is called a generator for ξ if $\zeta^- = \xi$.

There exists a unique partition $\pi(T)$ ($T\pi(T) = \pi(T)$) satisfying the following equivalent conditions ([9]):

$$h(T_\xi) = 0 \quad \text{if and only if} \quad \xi \leq \pi(T); \tag{1}$$

$$\text{if } \xi \in Z, \text{ then } h(T, \xi) = 0 \quad \text{if and only if} \quad \xi \leq \pi(T). \tag{2}$$

A partition ξ is called **extremal** if $T\xi \geq \xi$, $\xi^+ = \varepsilon$, and

$$\bigwedge_k T^{-k}\xi = \pi(T).$$

A partition ξ is called **perfect** if it is extremal and $h(T) = h(T, \xi)$.

§ 2. **Basic properties of perfect partitions.**

Theorem (Rokhlin and Sinai ([1])). *For every automorphism there exists a perfect partition.*

Theorem (Sinai ([5,7])). *If ξ^1, ξ are perfect partitions, $\xi^1 \leq \xi$, $h(T) < \infty$, then there exists $\eta \in Z$ such that*

$$\xi^1 = \eta(\xi), \qquad \text{where} \qquad \eta(\xi) = \bigwedge_n (\eta^- \vee T^{-n}\xi).$$

Theorem (Sinai ([5,7])). *If ξ^1 is extremal, ξ is perfect, $\xi^1 \leq \xi$, $h(T) < \infty$, then ξ^1 is perfect.*

Theorem (Sinai ([4])). *If ξ^1 is extremal, ξ is perfect,*

$$\xi^1 \leq \xi, \qquad h(T) < \infty, \qquad \text{then} \qquad \bigwedge_n (T^{-n}\xi \vee \xi^1) = \xi^1.$$

Proof. It is clear that ξ^1 is perfect, and then there exists $\eta \in Z$ such that $\xi^1 = \eta(\xi)$, i.e.

$$\xi^1 \leq \bigwedge_n (T^{-n}\xi \vee \xi^1) = \bigwedge_n \left(T^{-n}\xi \vee \bigwedge_m (T^{-m}\xi \vee \eta^-) \right) \leq$$

$$\leq \bigwedge_n (T^{-n}\xi \vee T^{-n}\xi \vee \eta^-) = \xi^1.$$

§ 3. **Basic theorems.** We shall need the following generalization of the results of ([6]).

Theorem 1. *If the automorphism T is ergodic, $T\xi \geq \xi \geq \eta \geq T\eta$, and the partition $\eta^1 \leq \xi$ is countable, then there exists a countable partition $\zeta \geq \eta^1$ such that*

$$\xi \geq \zeta^- \geq \eta \quad \text{and} \quad \zeta^+ \geq \xi \; (\zeta^+ \geq \xi^+).$$

Proof. We shall construct a countable partition ζ_1 such that

$$\eta^1 \leq \zeta_1 \leq \xi \quad (\zeta_1 \leq \xi) \quad \text{and} \quad \zeta_1^+ \geq \xi.$$

Taking T^{-1} instead of T, by the same method we construct a countable partition ζ_2 such that

$$\zeta_2 \leq \eta \quad (\zeta_2^+ \leq \eta,\ \zeta_2 \leq \xi) \quad \text{and} \quad \zeta_2^- \geq \eta.$$

Putting $\zeta = \zeta_1 \vee \zeta_2$, we shall have

$$\eta^1 \leq \zeta, \qquad \xi \geq \zeta^- \geq \eta, \qquad \zeta^+ \geq \xi.$$

Let $\eta^1 \leqslant \delta_0 \leqslant \delta_1 \leqslant \cdots$ be countable infinite partitions, with

$$\bigvee_n \delta_n = \xi.$$

Put

$$\xi_n = \delta_n \vee T^{-1}\delta_n \vee \cdots \vee T^{-n}\delta_n.$$

Then

$$\bigvee_n \xi_n = \xi$$

and $T^k\xi_n \geqslant \xi_{n-k}$. If $\delta_0 = \xi_0 = (A_0, A_1, \ldots)$, then there exists a sequence of natural numbers

$$a(0) < a(1) < \cdots$$

such that

$$\overline{\lim_{n\to\infty}} f(T^{-n}x) - n = +\infty \quad \text{(almost everywhere)},$$

where

$$f(x) = \sum_{i=0}^{\infty} a(i)\chi_{A_i}(x) \quad \text{(see (6))}.$$

Let ζ_1 be a countable partition coinciding on the set A_k with $\xi_{a(k)}$. It is clear that

$$\eta^1 \leqslant \zeta_1 \leqslant \xi.$$

By the method used in (6) one can show that

$$\zeta_1^+ \geqslant \xi.$$

Corollary. The partition ζ is a generator for the extremal partition ξ^-, if ξ is extremal and $\eta = \pi(T)$.

Theorem 2. If the automorphism T is ergodic, $h(T) < \infty$, and the partition ξ is perfect, then ξ has a countable generator.

Proof. Since $H(T\xi \mid \xi) < \infty$, it follows (see (2,7)) that there exists a partition $\eta^1 \in Z$ such that

$$\xi^\circ = \eta^1 \vee T^{-1}\xi.$$

By Theorem 1 there exists a countable partition ζ such that

$$\zeta \geqslant \eta^1, \qquad \zeta^+ \geqslant \xi, \qquad \xi \geqslant \zeta^- \geqslant \pi(T).$$

On the basis of the corollary to Theorem 1, the partition ζ^- is extremal and, by Sinai’ s theorem,

$$\bigwedge_n (\zeta^- \vee T^{-n}\xi) = \zeta^-.$$

But

$$\xi = \eta^1 \vee T^{-1}\eta^1 \vee \cdots \vee T^{-(n-1)}\eta^1 \vee T^{-n}\xi \leqslant \zeta^- \vee T^{-n}\xi \leqslant \xi.$$

Consequently,

$$\xi = \zeta^-.$$

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Note: Figure translations are in progress. See original paper for figures.