Doklady of the Academy of Sciences of the USSR

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

Doklady of the Academy of Sciences of the USSR
1967. Volume 173, No. 5

UDC 513.88:517.948.32

MATHEMATICS

G. E. KISELEVSKII

ON CYCLIC SUBSPACES OF DISSIPATIVE OPERATORS

(Presented by Academician I. N. Vekua, 10 VI 1966)

1. Let $\mathfrak{H}$ be a separable Hilbert space and let A be a bounded linear operator acting in $\mathfrak{H}$. If the operator A is unicellular ($^{1-5}$), then there exists for it a vector $g \in \mathfrak{H}$ such that the linear span of the sequence $g, Ag, A^2g, \ldots$ is dense in $\mathfrak{H}$. In the present article, for a certain class of operators, the converse assertion is established.

2. We shall agree to assign an operator A to the class Ω_0 when the following conditions are fulfilled: 1) the entire spectrum of the operator A is concentrated at zero; 2) the operator A is dissipative, i.e. its imaginary component

$$A_I = \frac{1}{2i}(A - A^*)$$

is nonnegative;

$$3) \qquad \sigma(A) = \overline{\lim_{\lambda \to 0} (|\lambda| \, |\ln |\lambda|| \, \|(A - \lambda E)^{-1}\|)} < \infty. \tag{1}$$

The class Ω_0 contains, in particular, all Volterra dissipative operators whose imaginary component is nuclear (^{3,6}).

If the subspace $\mathfrak{H}_0$ is invariant with respect to an operator $A \in \Omega_0$ and $A_0 = A|_{\mathfrak{H}_0}$ is the restriction of the operator A to the subspace $\mathfrak{H}_0$, then A_0 also belongs to Ω_0.

3. Let $\mathfrak{G}$ be a separable Hilbert space with scalar product $(f, g)_{\mathfrak{G}}$ $(f, g \in \mathfrak{G})$, and let B be a simple Volterra dissipative operator with one-dimensional imaginary component, defined in $\mathfrak{G}$. Consider the Hilbert space $\widetilde{\mathfrak{G}}$, consisting of sequences

$$\tilde{f} = \|f_1, f_2, \ldots\| \qquad \left(f_k \in \mathfrak{G}, \ \sum_{k=1}^{\infty} (f_k, f_k)_{\mathfrak{G}} < \infty\right),$$

in which the scalar product is defined by the equality

$$(\tilde{f}, \tilde{g})_{\widetilde{\mathfrak{G}}} = \sum_{k=1}^{\infty} (f_k, g_k)_{\mathfrak{G}},$$

and define in $\widetilde{\mathfrak{G}}$ the operator $\widetilde{B} = \widetilde{B}_l$ by the formula

$$\widetilde{B}\tilde{f} = \|Bf_1, Bf_2, \dots\| \qquad (l = \operatorname{sp} B_I). \tag{2}$$

Obviously, $B \in \Omega_0$ and $\sigma(\widetilde{B}) = 2l$.

The operator $\widetilde{B}$ has the following universal property*: every simple operator A of the class Ω_0, for which $\sigma(A) \leq 2l$, is unitarily equivalent to the restriction of the operator $\widetilde{B}_l$ to one of its invariant subspaces.

4. Denote by $\widetilde{\mathfrak{G}}_\xi$, where $\xi = (\xi_1, \xi_2, \dots)$ is an arbitrary orthonormal basis of the Hilbert space $l^{(2)}$, the subspace in $\widetilde{\mathfrak{G}}$ consisting of all vectors of the form $\|\xi_1 f, \xi_2 f, \dots\|$ $(f \in \mathfrak{G})$. The orthoprojector $\widetilde{P}_\xi$ onto the subspace $\widetilde{\mathfrak{G}}_\xi$ is defined by the formula

$$\widetilde{P}_\xi \tilde{f} = \|\xi_1 f, \xi_2 f, \dots\| \quad \left(\tilde{f} = \|f_1, f_2, \dots\|,\ f = \sum_{k=1}^{\infty} \bar{\xi}_k f_k\right). \tag{3}$$

The orthogonal complement $\widetilde{\mathfrak{G}}'_\xi = \widetilde{\mathfrak{G}} \ominus \widetilde{\mathfrak{G}}_\xi$, as is easy to see, is the set of vectors of the form

$$\|\varphi_1, \varphi_2, \dots\| \quad \left(\varphi_k \in \mathfrak{G},\ \sum_{k=1}^{\infty} \bar{\xi}_k \varphi_k = 0\right).$$

The following propositions are easily verified.

I. For any ξ, the subspaces $\widetilde{\mathfrak{G}}_\xi$ and $\widetilde{\mathfrak{G}}'_\xi$ are invariant with respect to the operator $\widetilde{B}$.

II. The operator $\widetilde{B}_\xi = \widetilde{B}|\widetilde{\mathfrak{G}}_\xi$ is simple, Volterra, dissipative; its imaginary component is one-dimensional and its trace is equal to l.

III. If $\xi^{(1)}, \xi^{(2)}, \dots$ is a complete orthonormal sequence in $l^{(2)}$, then the decomposition

* This assertion was originally established by M. S. Brodskii and the author (^{7}) (see also (^{8})) for the case when the imaginary component of A is nuclear, and then generalized by L. E. Isaev.

$$\widetilde{\mathfrak{G}} = \widetilde{\mathfrak{G}}_{\xi^{(1)}} \oplus \widetilde{\mathfrak{G}}_{\xi^{(2)}} \oplus \cdots \tag{4}$$

is valid (see (7)).

Theorem 1. Let $\widetilde{\mathfrak{G}}_0$ be a subspace invariant with respect to the operator $\widetilde{B}$, and let $\widetilde{B}_0 = \widetilde{B}|\widetilde{\mathfrak{G}}_0$. If the operator $\widetilde{B}_0$ is nonunivalent, then the subspace $\widetilde{\mathfrak{G}}$ has a nonzero intersection with each of the subspaces $\widetilde{\mathfrak{G}}'_{\xi}$.

5. Consider the Hilbert space $\vec{L} = L^{(2)}(0, l)$ of vector-functions

$$\mathbf{f} = \| f_1(x), f_2(x), \dots \| \quad \left(f_k(x) \in L^{(2)}(0, l) \right)$$

with scalar product

$$(\mathbf{f}, \mathbf{g}) = \sum_{k=1}^{\infty} \int_0^l f_k(x) \overline{g_k(x)} \, dx \tag{5}$$

and the integration operator acting in it

$$I\mathbf{f} = \| If_1(x), If_2(x), \dots \| \quad \left(If(x) = 2i \int_0^x f(t) \, dt \right).$$

According to a theorem of M. S. Livshits (^{9}), the operators $\widetilde{B}_{\xi}^{(k)} = \widetilde{B}|\widetilde{\mathfrak{G}}_{\xi^{(k)}}$ $(k = 1, 2, \dots)$ and I are unitarily equivalent. Therefore there exists an isometric mapping U_k of the subspace $\widetilde{\mathfrak{G}}_{\xi^{(k)}}$ onto the subspace $\vec{L}_k = \{\|0, \dots, f_k(x), 0, \dots\|\}$ such that

$$\widetilde{B}_{\xi^{(k)}} = U_k^* I_{kU} k \quad \left(I_k = I|\vec{L}_k, \ k = 1, 2, \dots \right),$$

$$\widetilde{B} = U^* I U \quad \left(U = \sum_{k=1}^{\infty} U_k \widetilde{P}_{\xi^{(k)}} \right).$$

6. Let $\mathfrak{H}_0$ be an invariant subspace of the operator A. We shall agree to call the subspace $\mathfrak{H}_0$ **cyclic** with respect to the operator A if there exists a vector $g \in \mathfrak{H}_0$ such that the closure of the linear span of the sequence $A^n g$ $(n = 0, 1, \dots)$ coincides with $\mathfrak{H}_0$. In this case we shall call the vector g a **generating vector** of the operator A in the subspace $\mathfrak{H}_0$.

In the paper (^{2}) it was proved that the function $g(x)$ is generating for the operator I in the space $L^{(2)}(0, l)$ if and only if the measure of the set of all $x \in [0, \delta]$ for which $g(x) \neq 0$ is positive for every $\delta > 0$.

Lemma. Let $\mathfrak{H}_1, \mathfrak{H}_2, \ldots$ be a sequence of invariant subspaces of a unicellular operator A, and let g_k be a generating vector of the operator A in the subspace $\mathfrak{H}_k$ $(k = 1, 2, \ldots)$, with

$$\sum_{k=1}^{\infty} (g_k, g_k) < \infty.$$

Then there exists a sequence of numbers $\xi_1, \xi_2, \ldots$

$$\left(\sum_{k=1}^{\infty} |\xi_k|^2 = 1 \right),$$

such that the vector

$$g = \sum_{k=1}^{\infty} \xi_k g_k$$

will be generating for the operator A in the subspace

$$\mathfrak{H} = \bigcup_{k=1}^{\infty} \mathfrak{H}_k.$$

Theorem 2. Let A be a simple operator of class Ω_0, and let $\mathfrak{H}_0$ be its invariant subspace. In order that the subspace $\mathfrak{H}_0$ be cyclic with respect to the operator A, it is necessary and sufficient that the restriction $A_0 = A|\mathfrak{H}_0$ be unicellular.

Proof. Sufficiency is obvious. Let us proceed to establish necessity.

Put $l = \frac{1}{2}\sigma(A_0)$. Using the proposition given in Sec. 3, we find an invariant subspace $\widetilde{\mathfrak{G}}_0$ with respect to the operator $\widetilde{B} = \widetilde{B}_l$ such that the operators A_0 and $\widetilde{B}_0 = \widetilde{B}|\widetilde{\mathfrak{G}}_0$ are unitarily equivalent. By the necessity condition, some vector $\tilde{g} = \|g_1, g_2, \ldots\|$ is generating for the operator $\widetilde{B}$ in the subspace $\widetilde{\mathfrak{G}}_0$. Each of the subspaces $\mathfrak{G}_k$, spanned respectively by the vectors $g_k, Bg_k, Bg_k^2, \ldots$ $(k = 1, 2, \ldots)$, is invariant with respect to the operator B. Since the operator B is unicellular (^{2}), applying the lemma we choose a sequence of numbers

$$\xi_1, \xi_2, \ldots \left(\sum_{k=1}^{\infty} |\xi_k|^2 = 1 \right)$$

so that the vector

$$g = \bar{\xi}_1 g_1 + \bar{\xi}_2 g_2 + \cdots$$

is generating for the operator B in the subspace

$$\mathfrak{G} = \bigcup_{k=1}^{\infty} \mathfrak{G}_k.$$

We shall show that $\mathfrak{G}_0 = \mathfrak{G}$. Indeed, assuming the contrary, we obtain that $\sigma(\widetilde{B}_0) < 2l$, and this contradicts the definition of l, in view of the unitary equivalence of the operators $\widetilde{B}_0$ and A_0.

Consider the decomposition (4) of the space $\widetilde{\mathfrak{G}}$, where $\xi^{(1)} = (\xi_1, \xi_2, ...)$, and $\xi^{(2)}, \xi^{(3)}, ...$ are arbitrary unit vectors supplementing $\xi^{(1)}$ to an orthonormal basis in $l^{(2)}$, and let U be an isometric mapping of the space $\widetilde{\mathfrak{G}}$ onto the space $\widetilde{L}$, defined in Sec. 5. The vector $U\tilde{g} = \dot{g} = \|g_1(x), g_2(x), ...\|$ will evidently be generating for the operator I in the subspace $\widetilde{L}_0 = U\widetilde{\mathfrak{G}}_0$. Since, according to formulas (3) and (6),

$$\tilde{g}^{(1)} = \widetilde{P}_{\xi^{(1)}}\tilde{g} = \|\xi_1 g, \xi_2 g, ...\|$$

and the vector g is generating for the operator B in $\mathfrak{G}$, it follows that $\tilde{g}^{(1)}$ is a generating vector of the operator $\widetilde{B}$ in $\widetilde{\mathfrak{G}}_{\xi^{(1)}}$. Therefore the function $g_1(x)$ will be generating for the operator I in the space $L^{(2)}(0, l)$.

Denote by $\mathfrak{M}$ the linear span of the sequence

$$I^n g = \|I^n g_1(x),\, I^n g_2(x), ...\| \quad (n = 0, 1, ...).$$

Since

$$I^n g(x) = \frac{(2i)^n}{(n-1)!}\int_0^x (x-t)^{n-1} g(t)\, dt = \frac{(2i)^n}{(n-1)!}\int_0^x t^{n-1} g(x-t)\, dt \quad (n = 1, 2, ...),$$

then $\mathfrak{M}$ is the linear span of the vectors $\mathbf{g}$ and $\|p * g_1, p * g_2, ...\|$, where $p * g$ denotes the convolution of the functions $p(x)$ and $g(x)$, while $p(x)$ ranges over the set of all polynomials $(0 \leq x \leq l)$. We shall show that the intersection of the subspaces $\mathfrak{M} = \vec{L}_0$ and $\vec{L}_1' = \vec{L} \ominus \vec{L}_1 = \{\|0, f_2(x), f_3(x), ...\|\}$ consists of the zero vector. To this end suppose that

$$(\lambda_n g_1 + p_n * g_1) \to 0, \qquad (\lambda_n g_k + p_n * g_k) \to h_k \quad (k = 2, 3, ...) \tag{7}$$

in the sense of convergence in $L^{(2)}(0, l)$. Using the known properties of convolution [(10)] and the equalities (7), we obtain $g_1 * h_k = 0$ $(k = 2, 3, ...)$. Taking into account that the function $g_1(x)$ is generating for the integration operator in

$L^{(2)}(0,l)$, and applying Titchmarsh' s theorem $^{(11)}$, we arrive at the conclusion that $h_k = 0$ $(k = 2, 3, ...)$. Thus, by Theorem 1, the operator I_0 is unicellular and, consequently, the operator A_0 is also unicellular.

Let us note that without the requirement of dissipativity the necessity in Theorem 2 is, generally speaking, false. Indeed, the operator $\mathbf{J} = \|If_1(x), -If_2(x)\|$, acting in the space $\vec{L} = \{\|f_1(x), f_2(x)\|\}$ $(f_k(x) \in L^{(2)}(0,l))$, is not unicellular. However, if the function $g(x)$ is generating for the operator I in $L^{(2)}(0,l)$, then the vector-function $\mathbf{g} = \|g(x), g(x)\|$ will be generating for the operator $\mathbf{J}$ in $\vec{L}$.

Zhitomir Pedagogical Institute
named after I. Franko

Received
8 VI 1966

REFERENCES

[1] M. S. Brodskii, DAN, **111**, No. 5 (1956).
[2] M. S. Brodskii, UMN, **12**, 2 (74), 129 (1957).
[3] M. S. Brodskii, M. S. Livshits, UMN, **13**, 1 (79), 3 (1958).
[4] L. A. Sakhnovich, DAN, **115**, No. 4 (1957).
[5] L. A. Sakhnovich, Izv. AN SSSR, ser. matem., **22**, 299 (1958).
[6] I. Ts. Gohberg, M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Moscow, 1965.
[7] M. S. Brodskii, G. E. Kiselevskii, Izv. AN SSSR, ser. matem., **30**, No. 6, 1212 (1966).
[8] B. Sz.-Nagy, C. Foiaş, Acta Sci. Math., Szeged, **26**, 3–4 (1965).
[9] M. S. Livshits, Matem. sborn., **34** (76), 145 (1954).
[10] I. M. Gelfand, D. A. Raikov, G. E. Shilov, *Commutative Normed Rings*, Moscow, 1960.
[11] E. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Moscow-Leningrad, 1948.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.