ON ESTIMATES OF THE RATE OF CONVERGENCE OF SOME PROJECTION METHODS FOR SOLVING ELLIPTIC EQUATIONS

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Abstract

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ON ESTIMATES OF THE RATE OF CONVERGENCE OF SOME PROJECTION METHODS FOR SOLVING ELLIPTIC EQUATIONS

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The purpose of the present work is to obtain new estimates of the rate of convergence for certain schemes of the method of nets, the method of lines, and the method of integral relations in the case of elliptic problems with a positive definite self-adjoint differential operator of order $2k$. For a one-dimensional fourth-order equation with discontinuous coefficients, estimates of the rate of convergence of the method of nets were obtained by A. A. Samarskii (^{1}) and Hao Shou (^{2}). V. I. Lebedev (^{3}) gave estimates, in the metric of the space W_2^k, for the rate of convergence of the method of nets for the polyharmonic equation with general boundary conditions, depending on the smoothness of the boundary of the domain.

The variational approach used here previously led ($^{4-9}$) to new estimates of the rate of convergence of the method of nets in the case of Dirichlet and Neumann problems for a second-order elliptic-type equation with measurable and bounded coefficients and with a square-summable right-hand side.

1°. Suppose that in the n-dimensional Euclidean space R_n there is a certain regular cellular subdivision, and let $\{\Omega^\nu\}$ be the collection of its n-dimensional cells, $\nu \in \mathfrak{N}$, where $\mathfrak{N}$ is some set of indices. Let $e_1, \dots, e_n$ be an orthonormal frame in R_n; k an integer; $D_{\rho_i}^{ke_i} u(x)$ the k-th right difference quotient in the direction of the i-th coordinate axis with step ρ_i ($\rho_i > 0,\ i = 1, \dots, n$):

$$D_{\rho_i}^{ke_i} u(x) = \underbrace{D_{\rho_i}^{e_i} \dots D_{\rho_i}^{e_i}}_{k} u(x), \qquad D_{\rho_i}^{e_i} u(x) = \frac{u(x + \rho_i e_i) - u(x)}{\rho_i}. \tag{1}$$

The set of points at which the function $u(x)$ must be evaluated in order to construct the difference derivative (1) will be called the scheme of $D_{\rho_i}^{ke_i}(x)$. By $u_{\rho^p}(x)$ we shall denote the p-fold averaging of the function u over the parallelepiped $0 < \xi_i < \rho_i,\ i = 1, \dots, n$:

$$u_{\rho^p}(x) = (u_{\rho^{p-1}})_\rho(x), \qquad p = 2, \dots, k; \qquad \rho = (\rho_1, \dots, \rho_n), \tag{2}$$

$$u_\rho(x) = (\dots((u_{\rho_1})_{\rho_2} \dots)_{\rho_n})(x), \qquad u_{\rho_i}(x) = \frac{1}{\rho_i} \int_0^{\rho_i} u(x + \xi e_i)\, d\xi.$$

We shall call the subdivision $\{\Omega^\nu\}$ regular if, for each interior point of any n-dimensional cell Ω^ν $(\nu \in \mathfrak{N})$, all points of the scheme $D_{\rho_i}^{ke_i}(x)$ $(i = 1, \dots, n)$ are interior points of some cells of the subdivision $\{\Omega^\nu\}$. In each n-dimensional cell Ω^ν we select a point x_0^ν so that the scheme $D_{\rho_i}^{ke_i}(x_0^\nu)$ consists only of points of the system $\{x_0^\nu\}$; the system of points $\{x_0^\nu\}$ will be called a net, and each point x_0^ν a node of the net. Let

$$h = \max_{\nu \in \mathfrak{N}} \operatorname{diam} \Omega^\nu.$$

In what follows we shall consider functions $u(x, \tau)$ defined in the direct product $R_n \times R_s$, $x \in R_n$, $\tau \in R_s$. Retaining the preceding constructions in the pro-

in the space R_n, consider the functions $v(x, \tau)$ that are piecewise constant with respect to R_n:

$$v(x, \tau) = v(x_0^\nu, \tau), \qquad x \in \Omega^\nu \subset R_n, \quad \nu \in \mathfrak{N}.$$

Let us average them by formula 2, regarding τ as a parameter:

$$\tilde{u}(x, \tau) = (v_\rho^k)(x, \tau), \qquad \rho = (\rho_1, \dots, \rho_n). \tag{3}$$

By $Q_{(\varepsilon)}$ we denote the ε-neighborhood of the domain $Q \subset R_n \times R_s$, and by Q^h the set of points of the domain Q whose distance to its boundary ∂Q is greater than h. For functions $u \in \dot{W}_2^k(Q_{(\varepsilon)})$ we define the modulus of continuity by the formula

$$\omega_{W_2^k(Q)}(u, h) = \sup_{\|t\| < h} \|u(x + t) - u(x)\|_{W_2^k(Q)}, \qquad h < \varepsilon / \sqrt{n + s}.$$

Theorem 1. Let the function $u \in W_2^k(Q_{(\varepsilon)})$, $Q = \Omega \times T$ (Ω and T are finite domains in the spaces R_n and R_s, respectively), and suppose that the partition $\{\Omega^\nu\}$ in R_n is regular. Then there exists a function $\tilde{u}(x, \tau)$ of the form (3) such that

$$\|u - \tilde{u}\|_{W_2^k(Q)} \leq C_1 \omega_{W_2^k(Q_{(\varepsilon/2)})}(u, a_1 \rho_0), \tag{4}$$

where $\rho_0 = \max_{i=1,\dots,n} \rho_i$, and the constants C_1, a_1 do not depend on u, ρ_0.

Consider the space $\bar{X}$ of functions $v(x_0^\nu, \tau) \in W_2^k(T)$, $\nu \in \mathfrak{N}$, continuous with respect to τ, defined on the direct product $\{x_0^\nu\} \times T$ and equal to zero for $x_0^\nu \in \Omega^q$, $q = k/\sqrt{n+s}\rho_0$, and also vanishing inside some neighborhood of the complement CT of the domain T to the whole space R_s. Let X be the isomorphic space of functions of the form (3), $v \in X$. By Q_t we denote the set $\{y, y - t \in Q\}$, $t \in R_n \times R_s$.

Suppose that the domain Q satisfies the following condition:

$(*)$ There exist $\widetilde{N}$ unit vectors $m_1, \dots, m_{\widetilde{N}}$ belonging to the $(n+s)$-dimensional space $R_n \times R_s$, as well as a positive number h_0 and a function $\varphi(h) > 0$, $\varphi(h) \to 0$ as $h \to 0$, such that for $h < h_0$ the boundary strip of width h lies in the union

$$\bigcup_{i=1}^{\widetilde{N}} (Q \setminus Q_{\varphi(h) m_i}).$$

Remark 1. Let C^2 be the class of piecewise twice continuously differentiable surfaces without zero angles. If $\partial Q \in C^2$, then condition $(*)$ is fulfilled for $\varphi(h) = k'h$, $k' = \text{const} > 0$.

Theorem 2. Let the function $u \in \dot{W}_2^k(Q)$, and let the domain Q satisfy condition $(*)$. Then there exists in the space $\bar{X}$ a function $v(x_0^\nu, \tau)$ such that the function $\tilde{u}(x, \tau)$ obtained by formula (3) belongs to the class $\dot{W}_2^k(Q)$ and the inequality

$$\|u - \tilde{u}\|_{W_2^k(Q)} \le C_0 \omega_{W_2^k(Q)}(u', \varphi(a_0 \rho_0)), \tag{5}$$

holds, where u' is the extension by zero of the function u outside the domain Q.

Theorem 3. Let $\tilde{u} \in \dot{W}_2^k(Q)$, $u(x, \tau) \in W_2^k(\Omega)$, $\tau \in T$. Then, if condition $(*)$ is fulfilled with respect to the domain Ω in the space $\bar{X}$, there exists a function $v(x_0, \tau)$ such that the function $\tilde{u}(x, \tau)$ obtained by formula (3) belongs to $\dot{W}_2^k(\Omega)$, $\tau \in T$, and the inequality

$$\|u(\tau) - \tilde{u}(\tau)\|_{W_2^k(\Omega)} \le C_0 \omega_{W_2^k(\Omega)}(u'(\tau), \varphi(a_0 \rho_0)),$$

holds, where $u'(\tau)$ is the extension by zero of the function $u(\tau)$ outside the domain Ω.

2°. The method of integral relations has been used repeatedly in practice ($^{7-11}$) and has yielded positive results, a detailed survey of which is contained in (^{12}). Some estimates of the rate of convergence of this method are given in (^{13}). Here, for brevity, we shall restrict ourselves only to the first boundary-value problem.

Let α and β be $(n+s)$-dimensional vectors: $\alpha = (\alpha_1, \dots, \alpha_{n+s})$, $\beta = (\beta_1, \dots, \beta_{n+s})$. Introduce the symbols

$$\alpha\beta = (\alpha_1\beta_1, \dots, \alpha_{n+s}\beta_{n+s});$$

$$e_0 = (\underbrace{1, \dots, 1}_{n}, \underbrace{0, \dots, 0}_{s}); \qquad \hat{e}_0 = (\underbrace{0, \dots, 0}_{n}, \underbrace{1, \dots, 1}_{s});$$

$$y = (x_1, \dots, x_n, \tau_1, \dots, \tau_s), \qquad x \in R_n, \quad \tau \in R_s; \qquad |\alpha| = \sum_{i=1}^{n+s} |\alpha_i|,$$

as well as the notation for derivatives, difference ratios, and averages:

$$D^\alpha = D^{\alpha_1 e_1} \dots D^{\alpha_{n+s} e_{n+s}}, \qquad D^{\alpha_i e_i} = \frac{\partial^{\alpha_i}}{\partial y_i^{\alpha_i}},$$

$$\overline{D}_{\rho_i}^{-e_i} u(y) = [u(y) - u(y - \rho_i e_i)]/\rho_i,$$

$$\overline{D}_{\rho_i}^{-\alpha_i e_i} = \underbrace{\overline{D}_{\rho_i}^{-e_i} \dots \overline{D}_{\rho_i}^{-e_i}}_{\alpha_i}, \qquad u_\rho^\alpha = (\dots (u_{\rho_1}^{\alpha_1})_{\rho_2}^{\alpha_2} \dots)_{\rho_{n+s}}^{\alpha_{n+s}}, \qquad y = (y_1, \dots, y_{n+s}),$$

$$u_{\hat{\rho}}^\alpha = (\dots (u_{\hat{\rho}_1}^{\alpha_1})_{\hat{\rho}_2}^{\alpha_2} \dots)_{\hat{\rho}_{n+s}}^{\alpha_{n+s}},$$

$$u_{\hat{\rho}_i}(y) = \frac{1}{\rho_i} \int_0^{\rho_i} u(y - \xi e_i)\, d\xi, \qquad i = 1, \dots, n+s.$$

Consider, in a bounded domain $Q = \Omega \times T$ of the space $R_n \times R_s$, the self-adjoint equation

$$A_0 u \equiv \sum_{i=0}^{k} (-1)^i \sum_{|\alpha|=|\beta|=i} D^\alpha a_\alpha^\beta D^\beta u = f(y), \qquad y \in Q. \tag{6}$$

Assume that the coefficients $a_\alpha^\beta(y)$ are measurable and bounded in the domain Q, and that $a_\alpha^\beta = a_\beta^\alpha$. Assume also that, for any numbers ξ_α, $|\alpha| = k$, the inequality

$$\sum_{|\alpha|=|\beta|=k} a_\alpha^\beta(y) \xi_\alpha \xi_\beta \geq \gamma \sum_{|\alpha|=k} \xi_\alpha^2, \qquad \gamma = \text{const} > 0 \tag{7}$$

holds. Let the quadratic form $[u,u]$,

$$[u,u_1] = \int_Q \sum_{i=0}^{k} \sum_{|\alpha|=|\beta|=i} a_\alpha^\beta D^\alpha u D^\beta u_1\, dy, \tag{8}$$

be positive definite, $[u,u] \geq \mu_0(u,u)$, where $(u,u_1) = \int_Q uu_1\, dy$. By a solution of the first boundary-value problem for equation (6) we shall mean [14] the solution of the problem of minimizing the quadratic functional $F(u) = [u,u] - 2(u,f)$ on the space $W_2^k(Q)$. As an approximate problem we consider the problem of minimizing the aforementioned functional on the space $\overline{X}$. Sufficient conditions for the minimum of $F(u)$ on $\overline{X}$ have the form

$$\overline{A}_0 v \equiv \sum_{i=0}^{k} \sum_{|\alpha|=|\beta|=i} (-1)^{|\alpha|} \overline{D}_{\rho\alpha}^{\alpha e_0} D^{\alpha \hat{e}_0} \left(\int_{\Omega^\nu} d_\alpha^\beta(x,\tau) D_{\rho\beta}^{\beta e_0} D^{\beta \hat{e}_0} v_{\rho k e_0 - \alpha e_0}\, dx \right)_{\hat{\rho} k e_0 - \alpha e_0} (x_0^\nu, \tau) = f_{\hat{\rho} k e_0}(x_0^\nu, \tau), \qquad v(x \tag{9}$$

The operator $\overline{A}_0$ is symmetric positive definite and can be extended to a self-adjoint operator in the sense of Friedrichs (we assume that the sta-

the scalar product in the space $\overline{X}$ induced by the space $\mathring{X} \subset L_2(Q)$, so that the isomorphism (3) is an isometry).

Theorem 4. *Under the assumptions formulated with respect to the first boundary-value problem for equation (6), the method of integral relations (9) converges, and the difference between the solution u^* of the exact problem (6) and the solution v_* of the approximate problem (9) is estimated by the inequality*

$$\|\tilde{u}_* - u^*\|_{W_2^k(Q)} \leq C_0' \, \omega_{W_2^k(Q)}(u^*, a_0 \varphi(\rho_0)), \tag{10}$$

where $\tilde{u}_* = (v_*)^{\rho_k}$, $u^{*'}$ is the extension by zero of the function u^* to the exterior of the domain Q, and the constants C_0', a_0 do not depend on u^*, ρ_0.

Corollary. *If $u^* \in W_2^{(k+1)}(Q)$, $\partial Q \in C^2$, then*

$$\|\tilde{u}_* - u^*\|_{W_2^k(Q)} \leq C_0'' \rho_0^{1/2}, \tag{11}$$

where C_0'' does not depend on the parameter ρ_0.

Remark 2. The results extend to the Neumann problem, to the corresponding problems for a strongly elliptic system of differential equations, to certain non-self-adjoint problems (see, for example, (7)), and also to certain nonlinear problems of the form $Pu = f$; here, with respect to the nonlinear operator P, it is necessary to require that the Gâteaux derivative P_u' satisfy the inequality

$$\nu_0\|g\|^2_{W_2^k(Q)} \leq (P'_u g, g) < \nu_1\|g\|^2_{W_2^k(Q)}, \qquad g \in W_2^k(Q),$$

uniformly in $u \in D(A)$; ν_i are constants independent of u (see (15), p. 317). In the case of the Neumann problem, for $u^* \in W_2^{(k+1)}(Q)$ the convergence has order ρ_0. For $s = 0$, system (9) is the method of nets for problem (6).

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Note: Figure translations are in progress. See original paper for figures.

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