

The stability of a certain third-order system with variable structure

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Abstract

The paper considers an automatic control system with a variable structure, described by a system of third-order differential equations, the parameters of which vary within wide limits. A cone located in the phase space of variables x_1 , x_2 , x_3 is taken as the switching surface. The switching law is selected depending on the sign of the quadratic form. In § 1, the conditions for the existence of sliding motion on the cone are determined. In § 2, a system of differential equations is derived, by virtue of which the sliding process on the cone occurs, and its stability is proven. In § 3, it is proven that the representative point of the phase space reaches the cone. The stability of the zero solution of the system of third-order differential equations is characterized by the representative point of the phase space reaching the cone and the subsequent stable motion along the cone. Bibliography: 10.

Full Text

Introduction

The study of variable structure systems has been addressed in several foundational works [?, ?, ?, ?]. Consider a system of differential equations where the control law undergoes discontinuities on a switching surface. Specifically, let the system be defined as:

$$\dot{x} = y, \quad \dot{y} = -bx - ay - u$$

where $u = \text{sign } x(Ax + y)$. When $x = 0$ and $Ax + y \neq 0$, the phase trajectories cross the switching line. However, if $Ax + y = 0$, a sliding mode may occur. In this context, we consider a third-order system:

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -cx - by - az - u\end{aligned}$$

where $u = \text{sign } x(Ax + By + z)$. The switching surface is defined by the plane $Ax + By + z = 0$. As noted in [?], for a general n -th order system with state variables x_1, x_2, \dots, x_n , the control law is typically given by $u = \text{sign } x_1 \sum_{i=1}^n c_i x_i$. The objective of this study is to analyze the stability and dynamics of such systems under various parameter constraints.

§ 1. System Stability Conditions

Consider a linear system with a discontinuous control component:

$$\begin{aligned}\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \alpha b_1 x_1 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \alpha b_2 x_2 \\ \dot{x}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3\end{aligned}\tag{1.1}$$

where a_{ik} are constant coefficients, b_1, b_2 are control gains, and $\alpha = \pm 1$ is determined by the sign of the switching function σ . Let the switching surface be defined by the quadratic form:

$$\sigma = Ax_1^2 + Bx_2^2 - Cx_3^2 = 0\tag{1.3}$$

The stability of the equilibrium point depends on the behavior of the derivative $\dot{\sigma}$ in the vicinity of the surface $\sigma = 0$. To ensure the existence of a sliding mode, the following conditions must be satisfied:

$$\lim_{\sigma \rightarrow +0} \dot{\sigma} < 0, \quad \lim_{\sigma \rightarrow -0} \dot{\sigma} > 0\tag{1.9}$$

Substituting the system equations (1.1) into the derivative of (1.3), we obtain the conditions for the parameters A, B, C and the coefficients a_{ik} . Specifically, for $\alpha = 1$, the requirement $\dot{\sigma} < 0$ leads to a quadratic inequality in terms of the state variables x_1, x_2 .

Lemma 1.1

If the coefficients of the system (1.1) satisfy the conditions: 1. $|a_{11} - a_{33}| < b_1$ and $|a_{22} - a_{33}| < b_2$ 2. $Aa_{12} + Ba_{21} = 0$, $Aa_{13} - Ca_{31} = 0$, and $Ba_{23} - Ca_{32} = 0$ then there exists a region on the surface $\sigma = 0$ where a sliding mode is maintained, ensuring the convergence of the state trajectories to the origin.

§ 2. Analysis of the Sliding Mode

When the system enters a sliding mode on the surface M_0 defined by $\sigma = 0$, the motion is governed by a reduced-order system. Let $M_0(z_1, z_2, z_3)$ be a point on the switching manifold. The equations of motion in the sliding mode can be derived by considering the average behavior of the discontinuous control. Following the method described in [?], we project the dynamics onto the tangent plane of the switching surface.

The resulting trajectories in the sliding mode are characterized by the system:

$$\begin{aligned}\dot{x}_1 &= \frac{A(a_{33} - a_{11} + a_{22})x_1x_2 + Aa_{33}x_1^2}{\sqrt{Ax_1^2 + Bx_2^2}} \\ \dot{x}_2 &= \frac{B(a_{33} - a_{22} + a_{11})x_1x_2 + Ba_{33}x_2^2}{\sqrt{Ax_1^2 + Bx_2^2}}\end{aligned}\quad (2.4)$$

To analyze the stability of this reduced system, we employ a Lyapunov function candidate $V = Ax_1^2 + Bx_2^2$. Calculating the time derivative \dot{V} along the trajectories of (2.4), we find that if $a_{33} < 0$, then $\dot{V} < 0$, implying that the origin is asymptotically stable within the sliding manifold.

§ 3. Global Convergence and Boundary Conditions

The global behavior of the system (1.1) involves analyzing whether trajectories starting outside the switching surface $\sigma = 0$ eventually reach it. Let D be a region in the phase space where $\sigma < 0$. We seek to prove that any trajectory $M_0(x_1^0, x_2^0, x_3^0)$ will intersect the surface $\sigma = 0$ in finite time.

Using the transformation $t = \rho\tau$, where ρ is a small parameter, we can rewrite the dynamics of σ as:

$$\frac{d\sigma}{d\tau} = 2A(a_{11} + ab_1)x_1^2 + 2B(a_{22} + ab_2)x_2^2 - 2Ca_{33}x_3^2 \quad (3.3)$$

For $\alpha = -1$, if the gains b_1, b_2 are sufficiently large, the derivative $\dot{\sigma}$ remains positive for $\sigma < 0$, forcing the state towards the switching surface. Conversely, for $\sigma > 0$, setting $\alpha = 1$ ensures $\dot{\sigma} < 0$. This “pushing” effect from both sides of the surface $\sigma = 0$ guarantees the reachability of the sliding manifold.

In conclusion, the proposed control law with a quadratic switching surface provides robust stability for the third-order system. The conditions derived in Lemma 1.1 ensure both the existence of the sliding mode and the asymptotic stability of the equilibrium point. These results extend the classical theory of variable structure systems to a broader class of nonlinear switching manifolds.

Note: Figure translations are in progress. See original paper for figures.

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