ON A TRANSFORMATION OF OPERATOR SUMS

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Abstract

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ON A TRANSFORMATION OF OPERATOR SUMS

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Let $D = d/dt$ be the differentiation operator, $\theta_\alpha = t^\alpha D$, α any complex number,* and

$$\theta_{\alpha_k}\theta_{\alpha_{k-1}}\cdots\theta_{\alpha_1} \tag{1}$$

denote the superposition of the operators $\theta_{\alpha_1}, \theta_{\alpha_2}, \ldots, \theta_{\alpha_k}$ $(k = 1, 2, \ldots)$, and let G_n be the set of all functions of t differentiable n times.

The operator sum

$$\sum_{p=1}^{n} a_{n-p}(\theta_{\alpha_k}\theta_{\alpha_{k-1}}\cdots\theta_{\alpha_1})^p \tag{2}$$

with arbitrary constant coefficients a_{n-p} can be transformed to the form

$$\sum_{\nu=1}^{nk} Q(n, k, \nu, \alpha_i, t)D^\nu, \tag{3}$$

where $Q(n, k, \nu, \alpha_i, t)$ are certain completely determined functions. Let

$$Q(n, k, \nu, \alpha_i, t) = \sum_{p=\mathrm{E}\left(\frac{\nu+k-1}{k}\right)}^{n} C(k, \nu, p, \alpha_i)a_{n-p}t^{-\mu\nu}, \tag{4}$$

where

$$\mu = k - \sum_{i=1}^{k} \alpha_i,$$

$$C(k,\nu,p,\alpha_i) = \frac{1}{(\nu!)^2} \sum_{m=\nu}^{pk} \frac{m!}{(m-\nu)!} S(k,p,m,\alpha_i),$$

$$S(k,p,m,\alpha_i) = \sum_{r=0}^{m} \frac{(-1)^{r+pk} m!}{r!(m-r)!} N(p,k,\alpha_i,r),$$

$$N(p,k,\alpha_i,r) = \prod_{j=1}^{p} R(p,k,\alpha_i,r,j),$$

$$\begin{aligned} R(p,k,\alpha_i,r,j) = {} & [r + (p-j+1)\mu + \alpha_k][r + (p-j+1)\mu + \alpha_k + \alpha_{k-1} - 1] \\ & \times \cdots \times [r + (p-j+1)\mu + \alpha_k + \alpha_{k-1} + \cdots + \alpha_1 - (k-1)], \end{aligned}$$

$\mathrm{E}\left(\frac{\nu+k-1}{k}\right)$ is the integer part of the number $\frac{\nu+k-1}{k}$.

Theorem. On the set G_{nk}, the operators (2) and (3)–(4) are equivalent.

The proof is obtained by generalizing our method ([1]).

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* The function t^α, as a general power function, is generally multivalued, and it is necessary to fix some single-valued branch of it.

Relying on the theorem, one can solve the following problems.

Problem 1. A linear differential operator is given,

$$q_0(t)D^s + q_1(t)D^{s-1} + \cdots + q_{s-1}(t)D,$$

where $q_0(t), q_1(t), \ldots, q_{s-1}(t)$ are linear combinations of generalized power functions. It is required to determine the possibility (or impossibility) of representing it in the form of a polynomial in the operator (1).

This problem is of interest in connection with methods for integrating ordinary linear differential equations supplied by the operational calculus of V. A. Ditkin ([2]) and its generalizations.

Problem 2. Let σ be any complex number,

$$\delta(n,m) = \begin{cases} 0, & \text{if } n < m, \\ 1, & \text{if } n \geq m, \end{cases}$$

for $p = 1, 2, \ldots$ and $\xi = 1, 2, \ldots, (p+1)k$ the numbers $A_{\xi,p}$ satisfy the recurrence equation

$$A_{\xi,p+1} = \sum_{\nu=1}^{\min(\xi,pk)} \sum_{r=1}^{k} \frac{\delta(\nu+r,\xi)\, r!\, \Gamma(\nu-p\sigma+1)}{(\nu+r-\xi)!(\xi-\nu)!\, \Gamma(\xi-p\sigma-r+1)} A_{r,1} A_{\nu,p}$$

under the conditions $A_{r,1} = c_r$, where the c_r are arbitrarily prescribed constants with the sole restriction $c_k \neq 0$, and it is required to find $A_{\xi,p}$ in the form of some analytic expression.

In the solution of this problem an important role is played by $C(k,\nu,p,\alpha_i)$.

The Lah numbers ([3]) and the classical Stirling numbers of the second kind are special cases of $A_{\xi,p}$.

In conclusion, let us note the formula

$$(xy)^{-\lambda/2} e^{-y} I_\lambda(2\sqrt{xy}) = \sum_{k=0}^{n} \frac{L_k^\lambda(x)}{\Gamma(\lambda+k+1)}(-y)^k + \frac{(-y)^{n+1}}{\Gamma(\lambda+n+2)} e^{x-\xi^2} L_{n+1}^\lambda(\xi^2), \tag{5}$$

$$x \geq y \geq 0, \qquad \lambda \geq -\tfrac{1}{2}, \qquad \xi = \sqrt{x} + \theta\sqrt{y}, \qquad |\theta| < 1.$$

It can be obtained by relying on a result of B. M. Levitan ([4]) and on our representation ([5]) of the Laguerre polynomials $L_n^\lambda(x)$ by means of the operator (1) for $k = 2$, $\alpha_1 = 1 + \lambda$, $\alpha_2 = -\lambda$. Formula (5) is closely related to the well-known formula of N. Ya. Sonin ([6]) and differs by the remainder term.

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