ON SOME MULTIDIMENSIONAL INTEGRAL OPERATORS WITH HOMOGENEOUS KERNELS

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Abstract

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ON SOME MULTIDIMENSIONAL INTEGRAL OPERATORS WITH HOMOGENEOUS KERNELS

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In the present paper we study the properties of linearity and complete continuity of certain classes of integral operators with kernels homogeneous of order $-n$, where n is the dimension of the integral. In the one-dimensional case their linearity in L^p was established by Hardy ([3]), and in a number of spaces of types M and C—in ([4]); a number of theorems on the solvability of integral equations was also obtained ([2]). Multidimensional operators with kernels of the form $T(x,y) = |x|^{-\alpha}|x-y|^{\alpha-n}$, $0 < \alpha < n$, were studied in ([1]). Such operators arise from differential equations with singular coefficients.

Let x and y be points of the n-dimensional Euclidean space E_n. A measurable function $\theta(x,y)$ is called homogeneous of order $-n$ if, for all $t > 0$,

$$\theta(tx, ty) = t^{-n}\theta(x,y). \tag{1}$$

We pose the problem of investigating the operators

$$\theta\varphi = \int_D \theta(x,y)\varphi(y)\,dy, \qquad x \in D, \tag{2}$$

where D is an arbitrary bounded or unbounded domain in E_n containing the origin.

Adopting the usual notation for the Banach spaces L^p, M, C, we shall also consider the subclass M^0 (M^∞) of functions from M that are continuous at the point $x = 0$ ($x = \infty$). If $\Phi(x)$ belongs to one of the indicated classes, then by L^p_β, M_β, M^0_β (M^∞_β), and C_β we shall denote the corresponding isometric Banach spaces of functions (for more details see ([1]))

$$\varphi(x) = |x|^{-\beta}\Phi(x), \qquad \|\varphi\|_\beta = \|\Phi\|. \tag{3}$$

Theorem 1. Let $\theta(x,y)$ be homogeneous of order $-n$, satisfy the summability condition*

$$Q(\beta) = \int_{E_n} |\theta(j,u)|\,|u|^{-\beta}\,du < +\infty, \qquad j = (1,0,\dots,0), \tag{4}$$

and be invariant under arbitrary rotations γ of the space E_n in the sense that

$$\theta[\gamma(x),\gamma(y)] = \theta(x,y). \tag{5}$$

Then the operator $\theta\varphi$ is linear in $M_\beta(D)$, and if $x=0$ ($x=\infty$) is an interior point of D, then also in $M_\beta^0(D)$ ($M_\beta^\infty(D)$). If neither of the points $x=0$, $x=\infty$ is a boundary point for D, then $\theta\varphi$ is linear in $C_\beta(D)$. If, finally, in addition to (4), the condition

$$Q_1(\beta) = \int_{E_n} |\theta(v,j)|\,|u|^{\beta-n}\,du < +\infty, \tag{6}$$

is satisfied, then the operator is linear in $L^p_{\beta-n/p}(D)$ for every $p \geq 1$.

* Multidimensional singular integrals ([5]) do not satisfy this condition.

Proof.

1) **Linearity in M_β.** Introducing the function $\Phi(x)$ in accordance with (3) and putting $\theta\varphi = |x|^{-\beta}\Omega(x)$, we have

$$\Omega(x) = \int_D \left\{\frac{|x|}{|y|}\right\}^\beta \theta(x,y)\Phi(y)\,dy. \tag{7}$$

Let $u = \gamma_x^{-1}(y)$ be such a rotation of the coordinate system $oy_1y_2\dots y_n$ after which the first axis oy_1 lies on the straight line ox, and let $y = \gamma_x(u)$ be the inverse rotation; let $u = \delta_x(v)$ be the similarity transformation $u_i = |x|v_i$, $i = 1,2,\dots,n$, and $y = l_x(v)$, $l_x = \gamma_x\delta_x$, their composition. Changing variables $y = l_x(v)$ and using (5) and (1), we have

$$\Omega(x) = \int_{D_x} |v|^{-\beta}\theta(j,v)\Phi[l_x(v)]\,dv, \tag{8}$$

where $D_x = l_x^{-1}(D)$ and $\operatorname{mes} D_x = |x|^{-n}\operatorname{mes} D$. Hence it follows easily that $\|\Omega\|_M \leq Q(\beta)\|\Phi\|_M$, or $\|\theta\|_{M_\beta} \leq Q(\beta)$, and our first assertion is proved.

2) **Linearity in M_β^0.** If $x=0$ is an interior point of the domain D, then as $x \to 0$, $D_x \to E_n$.

Applying to (8) the procedure of subtracting $\Phi(0)$, it is not difficult to show that $\lim \Omega(x) = \Omega(0)$ exists and

$$\Omega(0) = q(\beta)\Phi(0), \qquad q(\beta) = \int_{E_n} |v|^{-\beta}\theta(j, v)\, dv. \tag{9}$$

An analogous property belongs to the point $x = \infty$, if it is interior.

3) **Linearity in C_β.** Let $x \to x_0$, $x_0 \in \overline{D}$, and $x_0 \neq 0, \infty$. Then

$$\Omega(x) - \Omega(x_0) = \Omega_1(x) + \Omega_2(x),$$

where

$$\Omega_1(x) = \left(\int_{D_x} - \int_{D_{x_0}} \right) |v|^{-\beta}\theta(j, v)\Phi[l_x(v)]\, dv,$$

$$\Omega_2(x) = \int_{D_{x_0}} |v|^{-\beta}\theta(j, v)\{\Phi[l_x(v)] - \Phi[l_x(v_0)]\}\, dv.$$

Since $\lim_{x\to x_0} \Phi[l_x(v)] = \Phi[l_x(v_0)]$, we have $\lim_{x\to x_0} \Omega_2(x) = 0$. Passing to $\Omega_1(x)$, we note that

$$\int_{D_x} - \int_{D_{x_0}} = \int_{(D_x - D_{x_0}) + (D_{x_0} - D_x)}.$$

Assuming additionally that the boundary of the domain D has zero measure, one can prove that as $x \to x_0$, $\mathrm{mes}(D_x - D_{x_0}) \to 0$ and $\mathrm{mes}(D_{x_0} - D_x) \to 0$. Therefore $\lim_{x\to x_0} \Omega_1(x) = 0$, and the continuity of $\Omega(x)$ is proved.

4) **Linearity in $L^p_{\beta - n/p}$.** Putting $\varphi(x) = |x|^{n/p-\beta}\Phi(x)$, $\theta\varphi = |x|^{n/p-\beta}\Omega(x)$, for $\Omega(x)$ we obtain the form (7) with exponent $\beta - n/p$ instead of β. Applying Hölder' s inequality, raising to the power p and integrating, we obtain

$$\|\Omega\|_{L^p} \leq Q^{1/p'}(\beta) Q_1^{1/p}(\beta)\|\Phi\|_{L^p}, \quad \text{or} \quad \|\theta\|_{L^p_{\beta-n/p}} \leq Q^{1/p'}(\beta) Q^{1/p}(\beta).$$

The theorem is proved.

Remark 1. If $x = 0$ is a boundary point of the domain D, then the operator does not act in $M^0_\beta(D)$ and $C_\beta(D)$. An analogous role of a special point is played by $x = \infty$, if the domain D is unbounded.

A large class of kernels satisfying condition (5) is constituted by

$$\theta(x,y) = h\{|x|, |y|, |x-y|, |x+y|, (x, y/|y|)\},$$

where h denotes an arbitrary function of 5 variables satisfying the summability condition (4). If, moreover, h does not change under interchange of the first two arguments, then (6) can be reduced to (4) by the transformation with inverse radius-vectors and, thus, linearity in $L^p_{\beta-n/p}$ will be ensured by only one summability condition. Kernels of this type include, for example, the previously studied $T(x,y)$ [1] (except for cases where special manifolds are present) and the kernel $\theta_0(x,y) = \rho^{-n}$, where $\rho^2 = |x|^2 + |y|^2$. Condition (4) for θ_0 will be fulfilled for all β, $0 < \beta < n$.

Relying on this, we prove

Theorem 2. *Let $P(x,y)$ be homogeneous of degree n, positive definite, i.e. $P(x,y) > 0$ for all x, y for which $\rho^2 = |x|^2 + |y|^2 > 0$, and continuous for the same x, y. Then the operator*

$$P\varphi = \int_D \frac{\varphi(y)\,dy}{P(x,y)}$$

is linear in $M_\beta(D)$ and $L^p_{\beta-n/p}(D)$ for all β, $0 < \beta < n$, and $p \geqslant 1$.

Theorem 3. *Let the singular point $x = 0$ be interior. If, in addition to the assumptions of Theorem 1, it is assumed that $q(\beta) \neq 0$, where $q(\beta)$ is given by formula (9), then the operator $\theta\varphi$ will not be completely continuous in M_β, M^0_β, and C_β. It will not be completely continuous in $L^p_{\beta-n/p}$, $p \geqslant 1$, if it is assumed that $q_1(\beta) \neq 0$, where*

$$q_1(\beta) = \int_{|u|\leqslant 1} |u|^{-\beta} \theta(j,u)\,du.$$

The proof of Theorem 3 in M_β, M^0_β, and C_β is based on property (9) and is analogous to the proofs of the corresponding theorems in [1], while for $L^p_{\beta-n/p}$ it is analogous to [6]. It is possible that the theorem is true without any assumptions of the type $q(\beta) \neq 0$, as was the case in the one-dimensional case [4] (see also [7]).

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Note: Figure translations are in progress. See original paper for figures.