STABLY SOLVABLE AND STABLY NILPOTENT GROUPS

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In the study of infinite groups there naturally arises the question of conditions for the existence in them of infinite proper subgroups, or the equivalent question of the structure of infinite groups all of whose proper subgroups are finite (O. Yu. Schmidt' s problem). Under certain additional restrictions this problem was solved in the works ([1−4]). In these works, what is essentially resolved is the question of conditions for the existence of infinite abelian subgroups in infinite groups satisfying one or another additional condition.

S. N. Chernikov posed to the author an analogous problem concerning conditions for the existence, in solvable (nilpotent) and generalized solvable groups, of proper infinite solvable (nilpotent) subgroups of one or another class of solvability (class of nilpotency). In the present paper we consider a more general question connected with this problem: conditions for the existence of infinite solvable (and infinite nilpotent) subgroups possessing a special property, which below is called stable solvability (stable nilpotency).

1. The **class of solvability** of a solvable group $\mathfrak{G}$ is the length of the series of successive commutants

$$\mathfrak{G}^{(0)} = \mathfrak{G} \supseteq \mathfrak{G}^{(1)} \supseteq \ldots \supseteq \mathfrak{G}^{(i)} \supseteq \mathfrak{G}^{(i+1)} \supseteq \ldots$$

of this group, i.e. the least natural number s such that $\mathfrak{G}^{(s)} = E$.

We shall say that an infinite solvable group $\mathfrak{G}$ of solvability class s is **stably solvable** if every infinite solvable subgroup of the group $\mathfrak{G}$ (in particular, the group $\mathfrak{G}$ itself) having solvability class s contains a proper infinite solvable subgroup of solvability class s.

Theorem 1. An infinite solvable group of solvability class s contains a proper subgroup of solvability class s.

Corollary 1. A solvable group of solvability class s, all of whose proper subgroups have solvability class less than s, is a finite group.

Corollary 2. If a torsion-free group $\mathfrak{G}$ is solvable, then any nonidentity subgroup of it (in particular, the group $\mathfrak{G}$ itself) is stably solvable.

For the study of conditions under which solvable groups contain stably solvable subgroups, some auxiliary propositions are needed.

Lemma 1. Every nonextremal* solvable group of solvability class s has a proper infinite subgroup of solvability class s.

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* A group is called extremal if it is a finite extension of an abelian group satisfying the minimal condition for subgroups (see ([5])).

Corollary. In order that an infinite solvable group be stably solvable, it is sufficient that it have no complete subgroups distinct from the identity.

For abelian periodic groups, the condition formulated in the corollary of Lemma 1 is also necessary. However, there exist nonabelian stably solvable periodic groups containing complete subgroups distinct from the identity.

Example. Let the group $\mathfrak{G}$ be a splitting extension of a cyclic group of order p and a quasicyclic p-group $\mathfrak{M}$. Then $\mathfrak{G}$ is a semidirect product $\mathfrak{G} = \mathfrak{A} \cdot \mathfrak{M}$, where $\mathfrak{A}$ is an infinite invariant elementary abelian p-group; the group $\mathfrak{G}$ is solvable and has derived length 2. We shall show that the group $\mathfrak{G}$ is stably solvable. Let $\mathfrak{H}$ be an arbitrary infinite nonabelian subgroup of the group $\mathfrak{G}$. If $\mathfrak{A} \cdot \mathfrak{H} \neq \mathfrak{G}$, then $\mathfrak{H}$ has no complete subgroups and, by the corollary of Lemma 1, is stably solvable. If, however, $\mathfrak{A} \cdot \mathfrak{H} = \mathfrak{G}$, then a nontrivial (since $\mathfrak{H}$ is nonabelian) normal divisor $\mathfrak{H} \cap \mathfrak{A}$ of the group $\mathfrak{H}$ must be infinite, for the factor group $\mathfrak{H}/\mathfrak{H} \cap \mathfrak{A}$ is quasicyclic and the center of the group $\mathfrak{G}$ is equal to the identity (see (6)). Since $\mathfrak{H}$ is nonabelian, $\mathfrak{H} \cap \mathfrak{A}$ is infinite, and $\mathfrak{H}/\mathfrak{H} \cap \mathfrak{A}$ is a quasicyclic group; in the group $\mathfrak{H}$ there is a true infinite nonabelian subgroup. Consequently, the group $\mathfrak{G}$ is stably solvable, although it has the complete subgroup $\mathfrak{M}$.

Lemma 2. An arbitrary nonextremal solvable group $\mathfrak{G}$ of derived length s has a stably solvable subgroup of the same derived length s.

Relying on Lemmas 1 and 2, one can prove the following assertion.

Theorem 2. An arbitrary nonextremal solvable group $\mathfrak{G}$ of derived length s has stably solvable subgroups of derived length t for any $t = 1, 2, \dots, s$. In particular, if a solvable group $\mathfrak{G}$ of derived length s has elements of infinite order, then it has stably solvable subgroups of derived length t for all $t = 1, 2, \dots, s$. If all elements distinct from the identity of a solvable group $\mathfrak{G}$ have infinite order, then every nonidentity subgroup of it is stably solvable (see Corollary 2 of Theorem 1).

Corollary 1. In order that all infinite solvable subgroups of derived length s of a group $\mathfrak{G}$ having subgroups of this kind be stably solvable, it is sufficient that all infinite solvable subgroups of derived length $s-1$ of the group $\mathfrak{G}$ be stably solvable.

Corollary 2. In order that all infinite solvable subgroups of a group $\mathfrak{G}$ having subgroups of this kind be stably solvable, it is necessary and sufficient that the group $\mathfrak{G}$ have no quasicyclic subgroups.

We note that if a group is stably solvable, then not all its subgroups are necessarily stably solvable. A contradicting example is furnished by the group constructed above in the example.

2. In this paragraph we consider the question of finding stably solvable subgroups in generalized solvable groups.

Theorem 3. If an RN^*-group $\mathfrak{G}$ contains a solvable subgroup of derived length s (possibly finite), then the group $\mathfrak{G}$ either contains a stably solvable subgroup of derived length s, or is extremal.

Corollary 1. If an RN^*-group $\mathfrak{G}$ is nonsolvable, then it has stably solvable subgroups of derived length s for all $s = 1, 2, \ldots$.

In Theorem 3 it is sufficient to assume that every countable subgroup of the group $\mathfrak{G}$ is an RN^*-group. Therefore we have

Corollary 2. If a locally nilpotent group contains a solvable subgroup of derived length s (this subgroup may also be fin-

finite), then the group either contains a stably soluble subgroup of solubility class s, or is extremal.

3. The **nilpotency class** of a nilpotent group $\mathfrak{G}$ is the length of the upper central series

$$\mathfrak{Z}_0 = E \subseteq \mathfrak{Z}_1 \subseteq \cdots \subseteq \mathfrak{Z}_i \subseteq \mathfrak{Z}_{i+1} \subseteq \cdots$$

of this group, i.e. the least natural number c such that $\mathfrak{Z}_c = \mathfrak{G}$.

We shall say that an infinite nilpotent group $\mathfrak{G}$ of nilpotency class c is **stably nilpotent** if every infinite nilpotent subgroup of nilpotency class c of the group $\mathfrak{G}$ (in particular, the group $\mathfrak{G}$ itself) contains a proper infinite nilpotent subgroup of nilpotency class c.

Theorem 4. *An infinite nilpotent group of nilpotency class c contains a proper subgroup of nilpotency class c.*

Corollary 1. *A nilpotent group of nilpotency class c, all proper subgroups of which have nilpotency class less than c, is a finite group.*

Corollary 2. *If a torsion-free group $\mathfrak{G}$ is nilpotent, then any nontrivial subgroup of it (in particular, the group $\mathfrak{G}$ itself) is stably nilpotent.*

To study the conditions under which a nilpotent group has stably nilpotent subgroups, several auxiliary propositions will be needed.

Lemma 3. *Every non-extremal nilpotent group of nilpotency class c has a proper infinite subgroup of nilpotency class c.*

Corollary. *In order that a periodic infinite nilpotent group be stably nilpotent, it is necessary and sufficient that the group contain no complete subgroups.*

Let us note that the last assertion does not always hold for nilpotent groups with elements of infinite order (for example: the additive group of rational numbers).

Lemma 4. *An arbitrary non-extremal nilpotent group of nilpotency class c has a stably nilpotent subgroup of nilpotency class c.*

Relying on Lemmas 3 and 4, one can prove the following assertion.

Theorem 5. *If a non-extremal locally nilpotent group has a subgroup of nilpotency class c, different from the identity (possibly finite), then it has a stably nilpotent subgroup of nilpotency class c. In particular, if a locally nilpotent group $\mathfrak{G}$ contains elements of infinite order and has a subgroup of nilpotency class c different from the identity (possibly finite), then the group $\mathfrak{G}$ has a stably nilpotent subgroup of nilpotency class c.*

Corollary. *In order that all infinite nilpotent subgroups of the group $\mathfrak{G}$ having subgroups of this kind be stably nilpotent, it is necessary and sufficient that the group $\mathfrak{G}$ have no quasicyclic subgroups.*

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Note: Figure translations are in progress. See original paper for figures.