ON SINGULAR CAUCHY AND TRICOMI PROBLEMS

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Abstract

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MATHEMATICS

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ON SINGULAR CAUCHY AND TRICOMI PROBLEMS

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Consider in the half-plane $s \geq 0$ the equation

$$z_{xx} + \frac{c}{x} z_x = z_{ss} + \frac{a}{s} z_s + b^2 z \qquad (a, b, c = \text{const}) \tag{1}$$

and call $z(x, s)$, $\bar{z}(x, s)$ solutions of the Tricomi problems for (1), if

$$z(x, 0) = \tau(x), \qquad z(x, x) = \varphi(x), \qquad \tau(0) = \varphi(0); \tag{2a}$$

$$\bar{z}_\eta(x, 0) = \nu(x), \qquad \bar{z}(x, x) = \psi(x), \qquad \eta = -\left(\frac{s}{1-a}\right)^{1-a}. \tag{2b}$$

As shown in ([1]), z and $\bar{z}$ can be written in quadratures if the Riemann function $v(x, s; x_0, s_0)$ and the Green-Hadamard resolvents $H(x, s; x_0, s_0)$, $\bar{H}(x, s; x_0, s_0)$ of problems (1), (2) are known. Introducing, instead of $v, H, \bar{H}$, the expressions $U = \Phi_0 v$, $V = \Phi_0 H$, $\bar{V} = \Phi_0 \bar{H}$, $\Phi_0 = x_0^{-c} s_0^{-a}$, we obtain:

1. Let $a = 2\beta$, $c = 2\mu$, $\pi\delta = \sin \pi\beta$, $4ss_0\omega = R^2$, $4xx_0\lambda = -R^2$, $4\rho = b^2R^2$, $R^2 = (x - x_0)^2 - (s - s_0)^2$. Then for $0 < \beta < 1$

$$U = \delta(xx_0)^{-\mu}(ss_0)^{-\beta} \int_0^1 \xi^{-\beta}(1-\xi)^{\beta-1}(1-\xi\omega)^{-\beta} Q_0(\xi)\, d\xi; \tag{3a}$$

$$Q_0(\xi) = \Xi_2[\mu, 1-\mu, \beta; \lambda(1-\xi), \rho(1-\xi)], \tag{3b}$$

where $\Xi_2(\alpha, \beta, \gamma; x, y)$ is Humbert' s function. Further, for all $\beta < 1$

$$V = \delta(xx_0)^{-\mu}\omega^{a-1}\left(\frac{2}{R}\right)^a \int_0^1 [\xi(1-\xi)]^{-\beta}\left(1 - \frac{\xi}{\omega}\right)^{\beta-1} Q_1(\xi)\, d\xi; \tag{4a}$$

$$Q_1(\xi) = \Xi_2[\mu, 1-\mu, \beta; \lambda(1-\xi/\omega), \rho(1-\xi/\omega)], \tag{4b}$$

and, with the same notation and for $\beta > 0$,

$$\bar{V} = \delta(xx_0)^{-\mu} \left(\frac{2}{R}\right)^a \int_0^1 [\xi(1-\xi)]^{\beta-1} \left(1-\frac{\xi}{\omega}\right)^{-\beta} \bar{Q}_1(\xi)\, d\xi; \tag{5a}$$

$$\bar{Q}_1(\xi) = \Xi_2[\mu, 1-\mu, 1-\beta; \lambda(1-\xi/\omega), \rho(1-\xi/\omega)]. \tag{5b}$$

From (3), (4), (5) it further follows that

$$U = (xx_0)^{-\mu}(ss_0)^{-\beta} \sum_{n=0}^{\infty} A_n \rho^n F_3(\beta, \mu, 1-\beta, 1-\mu, 1+n; \omega, \lambda), \tag{6a}$$

$$V = \varkappa(xx_0)^{-\mu}\omega^{a-1} \left(\frac{2}{R}\right)^a \times$$

$$\times \sum_{n=0}^{\infty} B_n \rho^n H_2(1-\beta-n, 1-\beta, \mu, 1-\mu, 2-a; \omega^{-1}, -\lambda), \tag{6b}$$

$$\bar{V} = \bar{\varkappa}(xx_0)^{-\mu} \left(\frac{2}{R}\right)^a \sum_{n=0}^{\infty} \bar{B}_n \rho^n H_2(\beta-n, \beta, \mu, 1-\mu, a; \omega^{-1}, -\lambda). \tag{6c}$$

Here $(n!)^2 A_n = 1,\ n!(\beta)_{nB}n = 1,\ n!(1-\beta)_n \bar{B}_n = 1$, and $\varkappa, \bar{\varkappa}$ are indicated in ([2]).

2. With the aid of (6b), (6c), for $\varphi(x) = \psi(x) \equiv 0$ we obtain

$$z = \varkappa(1-a)(2s)^{1-a} \int_0^{x-s} \tau(x_0) r^{a-2} \left(\frac{x_0}{x}\right)^\mu \Xi_2(\mu, 1-\mu, \beta; \lambda_1, \rho_1)\, dx_0, \tag{7a}$$

$$\bar{z} = \frac{1}{2}\varkappa[2(1-a)]^a \int_0^{x-s} \nu(x_0) r^{-a} \left(\frac{x_0}{x}\right)^\mu \Xi_2(\mu, 1-\mu, 1-\beta; \lambda_1, \rho_1)\, dx_0, \tag{7b}$$

where $4xx_0\lambda_1 = -r^2,\ 4\rho_1 = b^2r^2,\ r = \sqrt{(x-x_0)^2 - s^2}$. If, however, $\tau(x) = x^\alpha$ $(\alpha > 0)$, then

$$z^{(\alpha)} = Dx^{-(c+\alpha+1)} s^{1-a} r^{a+c+2\alpha} \Xi_2\left(\frac{\alpha}{2}+1, \frac{\alpha+1}{2}+\mu, \gamma; \frac{r^2}{x^2}, \frac{1}{4}b^2r^2\right), \tag{8}$$

$$r^2 = x^2 - s^2, \qquad \gamma = \alpha + \beta + \mu + 1, \qquad -c - 2\alpha < a < 1,$$

$$\Gamma(\gamma)\Gamma(-\nu)D = \Gamma\left(\frac{\alpha}{2} + 1\right)\Gamma\left(\frac{\alpha+1}{2} + \mu\right).$$

Since $z(x,s)$ and $\bar{z}(x,s)$ are connected by the equality

$$\bar{z}[x, s; a, b, c; \nu(x)] \equiv \eta z[x, s; 2-a, b, c; \nu(x)], \tag{9}$$

(8) and (9) give a convergent solution $\bar{z}^{(\alpha)}(x,s)$ of problem (2b) for $\nu(x) = x^\alpha$.
In the more general case when $\tau(x)$ and $\nu(x)$ are given by power series

$$\tau(x) = \sum_{m=0}^{\infty} A_{mx}^{m+\alpha}, \qquad \nu(x) = \sum_{m=0}^{\infty} \bar{A}_{mx}^{m+\alpha} \quad (\alpha = \text{const} > 0), \tag{10}$$

we arrive at expansions in the functions $z^{(\alpha)}$ and $\bar{z}^{(\alpha)}$:

$$z(x,s) = \sum_{m=0}^{\infty} A_{mz}^{(m+\alpha)}(x,s), \qquad \bar{z}(x,s) = \sum_{m=0}^{\infty} \bar{A}_m \bar{z}^{(m+\alpha)}(x,s). \tag{11}$$

Finally, denoting by $z_0(x,s)$ and $\bar{z}_0(x,s)$ the solutions of the Cauchy problems for (1):

$$z_0(x,0) = \tau(x), \qquad z_{0n}(x,0) = 0, \qquad \bar{z}_0(x,0) = 0, \qquad \bar{z}_{0n}(x,0) = \nu(x), \tag{12}$$

we obtain, for $\beta > 0$, $4(1+\xi t)\omega_0 = t^2(\xi^2 - 1)$, $t = s/x$, $4\sigma = b^2 s^2(\xi^2 - 1)$,

$$z_0(x,s) = \gamma \int_{-1}^{1} \tau(x + \xi s)(1 - \xi^2)^{\beta - 1}(1 + \xi t)^\mu Q(x, s; \xi)\, d\xi, \tag{13a}$$

$$\sqrt{\pi}\Gamma(\beta)\gamma = \Gamma(\nu + 1), \qquad \nu = \beta - \frac{1}{2}, \qquad Q = \Xi_2(\mu, 1-\mu, \beta; \omega_0, \sigma),$$

and when $\beta < 1$,

$$\sqrt{\pi}\Gamma(1-\beta)\bar{\gamma} = \Gamma(1-\nu), \qquad \bar{Q} = \Xi_2(\mu, 1-\mu, 1-\beta; \omega_0, \sigma):$$

$$\bar{z}_0(x,s) = \bar{\gamma}\eta \int_{-1}^{1} \nu(x + \xi s)(1 - \xi^2)^{-\beta}(1 + \xi t)^\mu \bar{Q}(x, s; \xi)\, d\xi. \tag{13b}$$

This time the holomorphic data (10) correspond to formulas (11), in which

$$z_0^{(\alpha)}(x,s) = x^\alpha \Xi_2\left(-\frac{\alpha}{2}, \frac{1-\alpha}{2} - \mu, 1+\nu; \frac{s^2}{x^2}, -\frac{1}{4}b^2 s^2\right); \tag{14a}$$

$$\bar{z}_0^{(\alpha)}(x,s) = x^\alpha \eta \Xi_2\left(-\frac{\alpha}{2}, \frac{1-\alpha}{2} - \mu, 1-\nu; \frac{s^2}{x^2}, -\frac{1}{4}b^2 s^2\right). \tag{14b}$$

3. In analogous operators one solves problems (2), (12) for the equation

$$z_{xx} + \frac{c}{x} z_x = z_{ss} + \frac{a}{s} z_s + b(1+\bar{r}^2)^{-2} z \qquad (a,b,c = \text{const}), \tag{15}$$

where $\bar{r}^2 = x^2 - s^2$. Namely, here (3)–(6) likewise determine $U, V, \bar{V}$, but now

$$Q_0 = F_3[\mu,\gamma,1-\mu,1-\gamma,\beta;\lambda(1-\xi),\rho(1-\xi)], \qquad (n!)^2 A_n = \Gamma_n; \tag{16a}$$

$$Q_1 = F_3[\mu,\gamma,1-\mu,1-\gamma,\beta;\lambda(1-\xi/\omega),\rho(1-\xi/\omega)], \qquad n!(\beta)_{nB} n = \Gamma_n. \tag{16b}$$

and $\overline{Q}_1,\ \overline{B}_n$ arise from $Q_1,\ B_n$ after replacing β by $1-\beta$, where in (16)

$$(1+\bar{r}^2)(1+\bar{r}_0^2)\rho = R^2, \qquad \bar{r}_0^2 = x_0^2 - s_0^2, \qquad 4\gamma(1-\gamma) = b, \qquad \Gamma_n = (\gamma)_n(1-\gamma)_n.$$

The solutions z and $\overline{z}$ of the Tricomi problems (2), (15) for $\varphi = \psi = 0$ have the form

$$z = \chi(1-a)(2s)^{1-a} \int_0^{x-s} \tau(x_0) r^{a-2} \left(\frac{x_0}{x}\right)^\mu F_3(\mu,\gamma,1-\mu,1-\gamma,\beta;\lambda_1,\rho_1)\,dx_0; \tag{17a}$$

$$\overline{z} = \frac{1}{2}\,\overline{\chi}[2(1-a)]^a \int_0^{x-s} \nu(x_0) r^{-a} \left(\frac{x_0}{x}\right)^\mu F_3(\mu,\gamma,1-\mu,1-\gamma,1-\beta;\lambda_1,\rho_1)\,dx_0, \tag{17b}$$

where $(1+\bar{r}^2)(1+x_0^2)\rho_1 = r^2$, and r, λ_1, χ and $\overline{\chi}$ are the same as in (7). With the aid of (3a), (16a) we further find for the Cauchy problems (12), (15) ($r_1 = |s^2 - (x-x_0)^2|$)

$$z_0 = \gamma_0 s^{1-a} \int_{x-s}^{x+s} \tau(x_0) r_1^{a-2} \left(\frac{x_0}{x}\right)^{\mu} F_3(\mu, \gamma, 1-\mu, 1-\gamma, \beta; \lambda_2, \rho_1)\, dx_0; \quad (18a)$$

$$\overline{z}_0 = \overline{\gamma}_0 \int_{x-s}^{x+s} \nu(x_0) r_1^{-a} \left(\frac{x_0}{x}\right)^{\mu} F_3(\mu, \gamma, 1-\mu, 1-\gamma, 1-\beta; \lambda_2, \rho_1)\, dx_0; \quad (18b)$$

$$4xx_0\lambda_2 = -r_1^2, \qquad \sqrt{\pi}\,\Gamma(\beta)\gamma_0 = \Gamma(1+\nu),$$

$$\sqrt{\pi}\,\Gamma(1-\beta)(1-a)^{1-a}\overline{\gamma}_0 = -\Gamma(1-\nu).$$

Other integral representations are obtained for $U,\ V,\ \overline{V}$ by substituting in (13) and (18) the initial data $\tau(x), \nu(x)$ of these functions on the line $s = 0$.

4. In terms of the series E_2, the Cauchy problem (12a) is solved for the more general, than (1), nonhomogeneous equation with a singular perturbing function:

$$u_{xx} + \frac{c}{x} u_x = u_{ss} + \frac{a}{s} u_s + \left(b^2 + \frac{k}{s^2}\right) u - \frac{k}{s^2}\tau(x) \qquad (k = \text{const}). \qquad (19)$$

Here, under the condition $\beta > \mu > 0$, we find

$$u_0(x, s) = \delta_1 \int_{-1}^{1} \tau(x + \xi s)(1 - \xi^2)^{\beta-1}(1 + \xi t)^{\mu} T(x, s; \xi)\, d\xi; \qquad (20a)$$

$$T = \delta_2 \int_0^1 \eta^{\mu-1}(1-\eta)^{\beta-\mu-1}(1 - \eta\omega_0)^{\mu-1} Q(\eta)\, d\eta, \qquad (20b)$$

$$Q(\eta) = E_2[p, q, \beta - \mu; (1 - \xi^2)(1 - \eta), \sigma(1 - \eta)], \qquad (20c)$$

where p and q are the roots of the equation $\rho^2 - \nu\rho + k/4 = 0$; $\omega_0, \sigma, t, \beta, \mu, \nu$ are the same as in (13a);

$$\sqrt{\pi}\,\Gamma(\beta)\delta_1 = \Gamma(p+1)\Gamma(q+1), \qquad \Gamma(\mu)\Gamma(\beta - \mu)\delta_2 = \Gamma(\beta).$$

From (20b) it follows that

$$T = \sum_{n=0}^{\infty} (p)_n (q)_n [(\beta)_n n!]^{-1} (1 - \xi^2)^n E_2(\mu, 1 - \mu, \beta + n; \omega_0, \sigma). \qquad (21)$$

When $\beta = 0$ ($\beta = 1$), $\mu = 0$ ($\mu = 1$), $b = 0$, (6) and (21) give Appell, Humbert, and Horn series with two arguments; while in the general case (6), (21) determine confluent hypergeometric functions of three variables, so that, for example, (6) may be written in the form

$$\overline{V} = \overline{\chi}(xx_0)^{-\mu}(2/R)^a H_2^{(3)}(\beta, \beta, \mu, 1-\mu, a; \omega^{-1}, -\lambda, \rho). \tag{22}$$

For the purposes of regular continuation of the series (6b), (6c), the following autotransformation is used:

$$H_2(a, \beta, \gamma, \delta, \varepsilon; x, y) = (1-x)^{-a} H_2\left[a, \varepsilon - \beta, \gamma, \delta, \varepsilon; \frac{x}{x-1}, y(1-x)\right], \tag{23a}$$

as well as the quadratic transformation

$$\begin{gathered} H_2(\alpha, \beta, \gamma, \delta; 2\beta; x, y) = \\ = \left(1 - \frac{x}{2}\right)^{-\alpha} H_7\left[\alpha, \gamma, \delta, \beta + \frac{1}{2}; \frac{x^2}{4(2-x)^2}, y\left(1 - \frac{x}{2}\right)\right], \end{gathered} \tag{23b}$$

by which H_2 can be transformed in each term of the series (6b), (6c).

The results obtained generate various basis representations for $z, \bar{z}, z_0, \bar{z}_0, u_0$. Thus, starting from (11a), (14a) and the addition theorem for $\Xi_2(\alpha, \beta, \gamma; x, y)$ with respect to γ and y, we find, for $\beta_0 = \beta_2 - \beta_1 > -n$, $b_0 = \sqrt{b_1^2 - b_2^2}$, $\nu_k = \beta_k - \frac{1}{2}$,

$$z(x, s; a_2, b_2, c) = \sum_{m=0}^{n-1} (-1)^m g_m(\beta_2)(b_0 s)^{2m} z(x, s; a_2 + 2m, b_1, c) + R_n; \tag{24a}$$

$$R_n = \delta \int_0^1 \xi^{a_1}(1-\xi^2)^{\beta_0+n-1} U_n(s, \xi)\, z(x, \xi s; a_1, b_1, c)\, d\xi. \tag{24b}$$

Here U_n is the normalized Lommel function:

$$(\beta_0)_n n! U_n = (b_0 s/2)^{2n} {}_1F_2\left[1, \beta_0 + n, n+1; \frac{1}{4} b_0^2 s^2 (1-\xi^2)\right], \tag{25}$$

$$m!(\nu+1)_m 2^{2m} g_m(\beta) = (-1)^m, \qquad \delta\Gamma(\beta_0)\Gamma(\nu_1+1) = 2\Gamma(\nu_2+1).$$

Comparing (12), (19) with the homogeneous problem (1), (12), we arrive at the equality

$$u(x,s;a_2,b,c,k)=\sum_{m=0}^{n-1}D_m z(x,s;a_2+2m,b,c)+\overline{R}_n; \tag{26a}$$

$$\overline{R}_n=\int_0^1 \xi^{a_1}(1-\xi^2)^{\beta_0+n-1}Q_n(\xi)\,z(x,\xi s;a_1,b,c)\,d\xi, \tag{26b}$$

$$Q_n(\xi)=\varkappa_n\,{}_3F_2(1,p_2+n,q_2+n;n+1,\beta_0+n;1-\xi^2), \tag{26c}$$

$$\varkappa_n\Gamma(\beta_0+n)\Gamma(\nu_1+1)n!=2\Gamma(\nu_2+n+1)n!D_n=2\Gamma(p_2+1)\Gamma(q_2+1)(p_2)_n(q_2)_n,$$

where $\beta_0=\beta_2-\beta_1>-n,\ a_1>0$. Other basis representations arise when substituting into (13a), (18a), (20a) the expansions $(2\alpha=a+1-n)$

$$\tau(x+\xi s)=\sum_{n=0}^{\infty}\frac{1}{(\nu)_n}\left(\frac{s}{2}\right)^n C_n^{\nu}(\xi)D_x^n z_0(x,s;a+2n,0,0); \tag{27a}$$

$$\tau(x+\xi s)=\sum_{n=0}^{\infty}\frac{1}{(a)_n}\left(\frac{s}{2}\right)^n C_n^{a}(\xi)D_x^n z_0(x,s;a+n,0,0), \tag{27b}$$

as well as, for the confluent case, the series (27a)

$$\tau(x+\xi s)=\sum_{n=0}^{\infty}\frac{s^n}{n!}H_n\left(\frac{\xi}{2}\right)D_x^n w(x,s^2),$$

where $w(x,s)$ is the solution of the Cauchy problem $w_{xx}=w_s,\ w(x,0)=\tau(x)$. Finally, note that the functions $V,\bar{V}$, which possess logarithmic singularities on the characteristics $R=0$, may be used as fundamental solutions in boundary-value problems for the corresponding equations (1) and (15) of elliptic type. In particular, they make it possible to generalize the mean-value theorems investigated earlier in [3].

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Note: Figure translations are in progress. See original paper for figures.