The classification of trajectories of a dynamical system with cylindrical phase space

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Abstract

A system of differential equations containing angular coordinates is considered. The phase space of such a system is cylindrical. Based on the behavior of trajectories on covering spaces, a classification of the trajectories of the system under consideration is introduced. Conditions for the absence of Poisson-stable motions and conditions for the boundedness of solutions are provided. Bibliography: 7 items.

Full Text

Introduction

This section examines the stability and qualitative behavior of systems of differential equations, building upon the foundational work of E. A. Barbashin and N. N. Krasovskii [?]. We consider a system of the form:

$$\frac{d\phi_i}{dt} = \Phi_i(\phi_1, \dots, \phi_m, x_1, \dots, x_n)$$
$$\frac{dx_j}{dt} = X_j(\phi_1, \dots, \phi_m, x_1, \dots, x_n)$$

where $i = 1, \dots, m$ and $j = 1, \dots, n$. The functions Φ_i and X_j are assumed to be periodic with respect to the variables ϕ_i with a period of 2π. Such systems frequently arise in the study of phase synchronization and multidimensional dynamical systems.

Stability Analysis and Lyapunov Functions

Following the methodology established in [?] and [?], we investigate the existence of a limit set $R(\phi_1, \dots, \phi_s)$ and the behavior of the system trajectories relative to this set. A critical component of this analysis is the construction of a Lyapunov function $v(\phi, x)$. As demonstrated by Barbashin [?], the existence of a negative definite derivative $\dot{v}$ along the trajectories of the system is a sufficient condition for the asymptotic stability of the equilibrium set.

In the context of the qualitative theory of differential equations [?], we define a region U such that for any initial condition within this region, the trajectory $p(t)$ remains bounded and approaches the invariant set R as $t \to \infty$. Specifically, if the derivative of the Lyapunov function satisfies $\dot{v} \leq 0$, the trajectories will converge to the largest invariant subset where $\dot{v} = 0$.

Convergence and Invariant Sets

The analysis of the limit set q for a trajectory $p(t_n)$ as $t_n \to \infty$ is central to understanding the long-term dynamics. According to the theorems presented in [?] (p. 358), if a trajectory is bounded, its limit set is non-empty, compact, and invariant. For the system under consideration, we denote the potential limit values as v_0. If $v(\phi) = v_0$ and $\dot{v} = 0$, the system reaches a steady state or a limit cycle within the set R.

Applying the criteria from Barbashin [?] and the global stability theorems in [?], we can establish conditions under which the set $v = 0$ is globally attracting. This is particularly relevant for systems where the matrix of coefficients $A = \{a_{jk}\}$ satisfies specific negativity conditions, ensuring that the energy-like function $w = -\sum x_i^2$ acts as a robust descriptor of the system' s dissipation.

Conclusion

The mathematical framework provided by the works of Barbashin, Krasovskii, and Lefschetz [?, ?, ?] allows for a rigorous treatment of non-linear oscillations and phase space stability. By defining appropriate Lyapunov functions $v(x)$ and analyzing the properties of the derivative $\dot{v}$, we can conclude that for the given class of periodic systems, the trajectories converge to the predicted invariant manifolds, provided the structural constraints on the functions Φ_i and X_j are satisfied.

References

1. Barbashin, E. A., Krasovskii, N. N. On the stability of motion in the large. *Doklady Akademii Nauk SSSR*, 1952.
2. Pontryagin, L. S. *Ordinary Differential Equations*. Moscow, 1959.
3. Lefschetz, S. *Differential Equations: Geometric Theory*. Moscow, 1961.
4. Barbashin, E. A. *Lyapunov Functions*. Moscow, Nauka, 1967.
5. Bakaev, Yu. N. *Systems of Phase Synchronization*. Moscow, 1954.
6. Barbashin, E. A. *Introduction to the Theory of Stability*. Moscow, Nauka, 1967.
7. Krasovskii, N. N. *Certain Problems in the Theory of Stability of Motion*. Moscow, 1959.

Figures

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CLASSIFICATION OF TRAJECTORIES OF A DYNAMIC SYSTEM WITH CYLINDRICAL PHASE SPACE

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1. Investigation of oscillations of pendulum systems (simple pendulum, system of coupled pendulums, double pendulum, etc.) as well as the in-investigation of the dynamics of electromechanical systems, inertial television synchronization systems, phase-locked loop systems [1], leads to the necessity of considering a system of differential equations of the form

$$\frac{d\varphi_i}{dt} = \Phi_i(\varphi_1, \ldots, \varphi_m, x_1, \ldots, x_n) \quad (i = 1, \ldots, m),$$
$$\frac{dx_j}{dt} = X_j(\varphi_1, \ldots, \varphi_m, x_1, \ldots, x_n) \quad (j = 1, \ldots, m), \tag{1}$$

where variables $\varphi_1, \ldots, \varphi_m$ are angular (phase) coordinates and functions Φ_i, X_j are periodic functions (with period 2π) of these coordinates, variables $x_1, \ldots, x_0$ are Euclidean coordinates.

Without loss of generality, it can be assumed that the period of all angular coordinates is the same and equal to 2π. This means that the physical state of the system under consideration, described by points of the form $(\varphi_1 + 2k_1\pi, \ldots, \varphi_m + 2k_m\pi, x_1, \ldots, x_n)$, where $k_1, \ldots, k_m$ are integers, is identical. Identifying all points of the indicated form, we obtain a cylindrical phase space $R(\varphi_1, \ldots, \varphi_m)$. This space can be geor-geometrically represented as a topological product of an m-dimensional torus and an n-dimensional Euclidean space of variables $x_1, \ldots, x_n$.

Euclidean space R of variables $\varphi_1, \ldots, \varphi_1, \ldots, \varphi_m, x_1, \ldots, x_n$ will be a covering space for the cylindrical for the cylindrical space $R(\varphi_1, \ldots, \varphi_m)$ (see, for example, [2] and [3]).

The cylindrical space $R(\varphi_1, \ldots, \varphi_m)$ can be obtained from the space R, if this latter is cut along the surfaces $\varphi_i = -\pi$, $\varphi_i = \pi$ $(i = 1, \ldots, m)$ and gluing of the obtained «strip» along the surfaces of the cut is carried out. It is clear that such folding can be carried out not an all folding tan be carried out not for all coordinates $\varphi_1, \ldots, \varphi_m$, but only for some part of these coordinates, for example, for coordinates $\varphi_r, \ldots, \varphi_s, \ldots, \varphi_s$; the cylindrical space obtained in this way will be denoted by the symbol $R(\varphi_r, \ldots, \varphi_s)$. Obviously, the space $R(\varphi_r, \ldots, \varphi_s)$ is also a covering for the space $R(\varphi_1, \ldots, \varphi_m)$.

Assuming that any conditions ensuring the existence and continuability of solutions of the system of equations (1) are fulfilled, we obtain a dynamic system on the phase space
$R(\varphi_1, \ldots, \varphi_m)$. This dynamic system induces in any of the covering spaces

Figure 1: Figure 1

can be continuously deformed into another. The number of distinct independent classes of closed paths is called the **connectivity number** of the manifold. The order of connectivity of the space $\mathbb{R}(\varphi_1, \ldots, \varphi_m)$ is equal to $m+1$, thus, the maximum number of homotopically independent limit cycles in the space $\mathbb{R}(\varphi_1, \ldots, \varphi_m)$ is equal to $m+1$.

Such a maximal system can be represented, for example, by a system consisting of a 0-cycle and φ_i-cycles, where $i = 1, \ldots, m$.

Thus, the classification of limit cycles proposed by us provides a finer partition into classes compared to the classification based on the concept of homotopy, since all cycles of class distinct from class (0) and classes (φ_i), $i = 1, \ldots, m$, will be homotopically dependent on the indicated above limit cycles. Nevertheless, our classification, distinguished by its simplicity, allows one to characterize the arrangement of a limit cycle on the considered cylindrical phase space more precisely. So, for example, on a two-dimensional torus $\mathbb{R}(\varphi_1, \varphi_2)$ we have three homomotopically independent classes of limit cycles: 0-cycles, φ_1-cycles, embracing the torus along the meridian, φ_2-cycles, embracing the torus along the parallel. Obviously, (φ_1, φ_2)-cycles will embrace the torus both along the meridian and along the parallel, i.e. despite the fact that they are derivatives of the cycles indicated above, they may be of independent interest during investigation.

3. As property A, any other property of the trajectories of a dynamical system can also be taken. One such property can serve, for example, the property of positive stability of trajectories according to Poisson ([5], p. 363).

Let us recall that point p is called positively stable according to Poisson, if for any neighborhood U of this point it is possible to indicate a positive number$T$ such that during its movement along the trajectory within the time interval $t \geq T$ the point p at least once enters the neighborbhood U again. It is known that if at least one point of a trajectory is positively stable according to Poisson, then all other points of this trajectory will possess the same property. Thus, we obtain the concept of a positively stable according to Poisson trajectory (P-stable trajectory).

Following definition from p. 2, we will say that a trajectory from the space $R(\varphi_1, \ldots, \phi_m)$ is P-stable of class $(\varphi_r, \ldots, \phi_s)$, if property thom-$P$-stability is preserved upon unrolling along all coordinates ϕ_i, where i is distinct from $r, \ldots, s$, and disappears upon further unrolling along any of the coordinates $\phi_r, \ldots, \phi_s$.

If as the space $R(\phi_1, \phi_2)$ we take a two-dimensional torus, then the only P-stable trajectories on it, distinct from rest points and limit cycles, will be P-stable trajectories of class (ϕ_1, ϕ_2). Indeed, upon unrolling the torus along any of the coordinates ϕ_1, ϕ_2 we get a cylinder, and along both coordinates ϕ_1, ϕ_2 – a plane. But neither on a plane, nor on a two-dimensional cylinder can there be P-stable trajectories, distinct from rest points and limit cycles ([5], p. 364).

Thus, on a torus there exist only P-stable trajectories of class (ϕ_1, ϕ_2), distinct from singular points and limit cycles. The property of P-stability of these trajectories is lost upon unrolling along any of the coordinates ϕ_1, ϕ_2.

If as the space $R(\phi_1, \phi_2)$ we take the topological product of a torus and a real line, i.e. introduce into consideration a new coordinate x, which is not

Figure 2: Figure 2

spaces will be three-dimensional cylinders, and the full development — a three-dimensional Euclidean space.

Obviously, in this case it is not negligible to construct examples of P-stolvile trajectories, different of entnu-singular tonks and limit cycles, of many of the future possible classes.

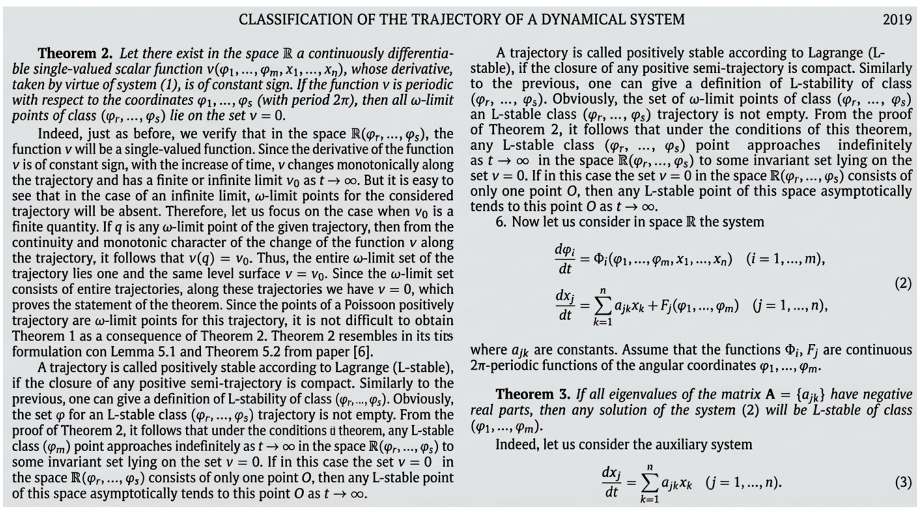
4. Let us now formulate one criterion for the absence of P-stolvile trajectories.

Theorem 1. Let there exist in the space R is equivicely continuous differentiable scalar function $v(\varphi_1, ..., \varphi_m, x_1, ..., x_n)$, whose positive, constant, then by virtue of system (1), is constant in space. Let the function $v(\varphi_1, ..., \varphi_m, x_1, ..., x_n)$ be periodic (in period 2π) in the coordinates $\varphi_1, ..., \varphi_s$. Then every P-stolvile trajectory of class $(\varphi_r, ..., \varphi_s)$ (a take of class (0) and class $(\varphi_p, ..., \varphi_q)$, where $\varphi_r, ..., (\varphi_p, ..., \varphi_q)$, where $\varphi_p, ..., \varphi_q$—coordinates from the snows $\varphi_r, ..., \varphi_s$) lewat on the snowest $v = 0$. Thus, if the snowest $v = 0$ commits any below trajectories, tho P-stolvile trajectories of the classes, indicated in the theorem are absent.

For the proof of theorem, let us consider the space $R(\varphi_r, ..., \varphi_s)$, which is onlyened from the space R by cladding along the coordinates $\varphi_r, ..., \varphi_s$. It suggest, that the function $v(\varphi_1, ..., \varphi_m, x_1, ..., x_n)$ is equally undefined and continuous in the space space $R(\varphi_r, ..., \varphi_s)$, since the same values of this function at corresponding points upon cladding filled by the same. Let us assume now, for definiteness, the the function $v(\varphi_1, ..., \varphi_m, x_1, x_1, ..., x_n)$ is non-positive in the space R, t.e. it demonstrates everyndere the inequality $v \leq 0$. Let us assume further, that there consider a P-stolvile trajectory of class $(\varphi_r, ..., \varphi_s)$. By definition, this trajectory first reaches P-stolvile in in space $R(\varphi_r, ..., R(\varphi_r, ..., \varphi_s)$ a belongs that property upon possessing any choice coordinate $\varphi_r, \ldots, \varphi_s$. Let us assume, that on our trajectory that is at menet odna point q at which v y—[illegible], au trans tdopian q at which v parameters of zero, t.e., t.e.,..., argumentatively. By the definition of P-stability, for any positive number T possible indicate number $t \geq T$ such, that after the interval time t point q reaches in a point t, m, that after the interval time t point q reaches in a given correctness of itself initial position. State estimation. Thus the function v will respecting our estimate somewhat trihemate in shown absolutely close k $v(q)$. the subsequent leads k contradiction, since function v – a nonincreasing function along the considered trajectory, a no choice point q showing that function v – in respecting our estimate byder known therefore, them the shown, proven by the function v at point q.

5. Let us now attempt to obtain a boree geliged result. We will again consider the four-dimensional space $R(\varphi_1, \ldots, \varphi_m)$. Consider positive semi-trajectory of some point p a sequence positions $p(t_n)$ that point upon movement along the trajectory system (1), corresponding the sequence of moments time $0 < t_1 < t_2 < \ldots, t_n \to \infty$. By definition, any point q, which is defined set $\{p(t_n)\}$, called an ω-limit point point p. As is known ([5], p. 338), the role of ω-limit points point of given point p is closed in invariant, t.. on consists ω-limit points ɐh, irom, its given point of nuch inme that is closed in invariant, t, t.e. on consists of entire trajectories. We cause the now, that point q is an ω-limit point of class $(\varphi_r, ..., \varphi_s)$ belongs point p, if point q is an ω-limit graph p in the space $R(\varphi_r, \ldots, \varphi_s)$ and now the property upon possessing any coordinates

Figure 3: Figure 3



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Theorem 2. *Let there exist in the space $\mathbb{R}$ a continuously differentiable single-valued scalar function $v(\varphi_1, ..., \varphi_m, x_1, ..., x_n)$, whose derivative, taken by virtue of system (1), is of constant sign. If the function v is periodic with respect to the coordinates $\varphi_1, ..., \varphi_s$ (with period 2π), then all ω-limit points of class $(\varphi_r, ..., \varphi_s)$ lie on the set $v = 0$.*

Indeed, just as before, we verify that in the space $\mathbb{R}(\varphi_r, ..., \varphi_s)$, the function v will be a single-valued function. Since the derivative of the function v is of constant sign, with the increase of time, v changes monotonically along the trajectory and has a finite or infinite limit v_0 as $t \to \infty$. But it is easy to see that in the case of an infinite limit, ω-limit points for the considered trajectory will be absent. Therefore, let us focus on the case when v_0 is a finite quantity. If q is any ω-limit point of the given trajectory, then from the continuity and monotonic character of the change of the function v along the trajectory, it follows that $v(q) = v_0$. Thus, the entire ω-limit set of the trajectory lies one and the same level surface $v = v_0$. Since the ω-limit set consists of entire trajectories, along these trajectories we have $v = 0$, which proves the statement of the theorem. Since the points of a Poissoon positively trajectory are ω-limit points for this trajectory, it is not difficult to obtain Theorem 1 as a consequence of Theorem 2. Theorem 2 resembles in its tits formulation con Lemma 5.1 and Theorem 5.2 from paper [6].

A trajectory is called positively stable according to Lagrange (L-stable), if the closure of any positive semi-trajectory is compact. Similarly to the previous, one can give a definition of L-stability of class $(\varphi_r, ..., \varphi_s)$. Obviously, the set φ for an L-stable class $(\varphi_r, ..., \varphi_s)$ trajectory is not empty. From the proof of Theorem 2, it follows that under the conditions ü theorem, any L-stable class (φ_m) point approaches indefinitely as $t \to \infty$ in the space $\mathbb{R}(\varphi_r, ..., \varphi_s)$ to some invariant set lying on the set $v = 0$. If in this case the set $v = 0$ in the space $\mathbb{R}(\varphi_r, ..., \varphi_s)$ consists of only one point O, then any L-stable point of this space asymptotically tends to this point O as $t \to \infty$.

A trajectory is called positively stable according to Lagrange (L-stable), if the closure of any positive semi-trajectory is compact. Similarly to the previous, one can give a definition of L-stability of class $(\varphi_r, ..., \varphi_s)$. Obviously, the set of ω-limit points of class $(\varphi_r, ..., \varphi_s)$ an L-stable class $(\varphi_r, ..., \varphi_s)$ trajectory is not empty. From the proof of Theorem 2, it follows that under the conditions of this theorem, any L-stable class $(\varphi_r, ..., \varphi_s)$ point approaches indefinitely as $t \to \infty$ in the space $\mathbb{R}(\varphi_r, ..., \varphi_s)$ to some invariant set lying on the set $v = 0$. If in this case the set $v = 0$ in the space $\mathbb{R}(\varphi_r, ..., \varphi_s)$ consists of only one point O, then any L-stable point of this space asymptotically tends to this point O as $t \to \infty$.

6. Now let us consider in space $\mathbb{R}$ the system

$$\frac{d\varphi_i}{dt} = \Phi_i(\varphi_1, ..., \varphi_m, x_1, ..., x_n) \quad (i = 1, ..., m),$$
$$\frac{dx_j}{dt} = \sum_{k=1}^{n} a_{jk} x_k + F_j(\varphi_1, ..., \varphi_m) \quad (j = 1, ..., n), \tag{2}$$

where a_{jk} are constants. Assume that the functions Φ_i, F_j are continuous 2π-periodic functions of the angular coordinates $\varphi_1, ..., \varphi_m$.

Theorem 3. *If all eigenvalues of the matrix $\mathbf{A} = \{a_{jk}\}$ have negative real parts, then any solution of the system* (2) *will be L-stable of class $(\varphi_1, ..., \varphi_m)$.*

Indeed, let us consider the auxiliary system

$$\frac{dx_j}{dt} = \sum_{k=1}^{n} a_{jk} x_k \quad (j = 1, ..., n). \tag{3}$$

Figure 4: Figure 4

By virtue of the well-known thorem of Lyapunov ([6], p. 35), there exists a positive definive quadratic form $v(x_1, \ldots, x_n)$, the derivative of which, taken by virtue of system (3), is equal to the function $w = -x_1^2 - \ldots - x_n^2$.

Taking the derivative of the function v, by virtue of system (2), we obtain

$$\frac{dv}{dt} = w + \sum_{j=1}^{n} \frac{\partial v}{\partial x_j} F_j.$$

Ctensider new in space $\mathbb{R}$ the cylinder $x_1^2 + \ldots + x_n^2 = r^2$. Since the functions F_j — are organved functions of arguments $\varphi_1, \ldots, \varphi_m$, then, choosing r sufficiently large, we obtain on the surface f the cylinder and outside it the nepancity of the colnmey and outside it the imagensity $v < -\varepsilon^2 < 0$. But this means that all trajectories of the system (2) going with increasing time inside the cylinder cemain there forever. Since the interier part of the cylinder passes into itself under the mapping defened by $\mathbb{R}$, for all angular coordinates in a organuunded set, we obtain, thus, in the space $R(\varphi_1, \ldots, \varphi_m)$ L-stability.

Corolliary. Let this system (2) exist as a continuously differentiable single-valued function $v(\varphi_1, \ldots, \varphi_m, x_1, \ldots, x_n)$ in the space $\mathbb{R}$. This snumr:tor R. Assume that the function is periodic (with period 2π) in all angular coordinates $\varphi_1, \ldots, \varphi_m$, and its derivative v, taken by virtue of system (2), is sign-definite. Iff all eigenvalues and ensures of the matritax $A = \{a_{lk}\}$ have negative real parts, then ewery trajectory of system (2) infinnevly approaches as $t \to \infty$ to an sheet as $t \to \infty$ to an invariant sheet, leying on the snowect $v = 0$.

In particular, if the snet $v = 0$ in the space $R(\varphi_1, \ldots, \varphi_m)$ contains as an invariant snet only one point, then upon fulfillment of the conditions stated above, we obtain unbaunded approximation of trajectories to this point.

However, this particular casse is met very rarely in applications. A more prevalent case is when there is some region of attraction of stable equilibrium position, this region is brounded by separating surfaces, passing through unstable equilibrium positions.

Literature

1. Bakaev Yu. N. Some questions of nonlinear theory of phase systems. Tr. Transactions of the Zhukovsky Air Force Engineering Academy, v. 800, M., 1950.
2. Pontryagin L. S. Continuous groups, 2nd ed. M., Fizmatgiz, 1954.
3. Seifert H., Threlfall W. Topology, GONTI, M.—L., 1938.
4. Barbashin E. A. Mathematical Collection, 29 (21), n. 3, 1951, pp. 501—518.
5. Nemytskii V. V., Stepanov V. V. Qualitative theory of differential equations, M.—L., GITTL, 1940.
6. Barbashin E. A. Introduction to the theory of stability, M., «Nauka», 1967.
7. LaSalle J., Lefschetz S. Investigation of stability by the direct method of Lyapunov. M., «Mir», 1964.

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Figure 5: Figure 5